NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 1360

THE STABILITY OF THE LAMINAR BOUNDARY LAYER IN A COMPRESSIBLE FLUID

By Lester Lees

Langley Memorial Aeronautical Laboratory
Langley Field, Va.

Washington
July 1947

TABLE OF CONTENTS

Page

Summary 1

Introduction 3

Symbols 3

1. Preliminary Considerations 6

2. Calculation of the Limits of Stability of the Laminar Boundary Layer in a Viscous Conductive Gas 11

3. Destabilizing Influence of Viscosity at Very Large Reynolds Numbers; Extension of Heisenberg's Criterion to the Compressible Fluid 21

4. Stability of Laminar Boundary Layer at Large Reynolds Numbers 27

a. Subsonic Free-Stream Velocity $\left(M_o < 1\right)$ 27

b. Supersonic Free-Stream Velocity $\left(M_o > 1\right)$ 32

5. Criterion for the Minimum Critical Reynolds Number 39

6. Physical Significance of Results of Stability Analysis 45

a. General 45

b. Effect of Free-Stream Mach Number and Thermal Conditions at Solid Surface on Stability of Laminar Boundary Layer 46

c. Results of Detailed Stability Calculations for Insulated and Noninsulated Surfaces 52

d. Instability of Laminar Boundary Layer and Transition to Turbulent Flow 53

7. Stability of the Laminar Boundary-Layer Flow of a Gas with a Pressure Gradient in the Direction of the Free Stream 56

Conclusions 59

Appendixes 64

Appendix A - Calculation of Integrals Appearing in the Inviscid Solutions 64

General Plan of Calculation 65

Detailed Calculations 68

Evaluation of $f_k^{(m)}$ 105

Order of Magnitude of Imaginary Parts of Integrals H_2, M_3, and N_3 110

Appendix B - Calculation of Mean-Velocity and Mean-Temperature Distribution across Boundary Layer and the Velocity and Temperature Derivatives at the Solid Surface 113

Mean Velocity-Temperature Distribution across Boundary Layer 113

Calculation of Mean-Velocity and Mean-Temperature Derivatives 119

Appendix C - Rapid Approximation to the Function $(1 - 2\lambda)v(c)$ and the Minimum Critical Reynolds Number 125

Appendix D - Behavior of $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ from Equations of Mean Motion 129

Appendix E - Calculation of Critical Mach Number for Stabilization of Laminar Boundary Layer 132

References 135

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE NO. 1360

THE STABILITY OF THE LAMINAR BOUNDARY LAYER IN A COMPRESSIBLE FLUID

By Lester Lees

SUMMARY

The present paper is a continuation of a theoretical investigation of the stability of the laminar boundary layer in a compressible fluid. An approximate estimate for the minimum critical Reynolds number $R_{\theta_{cr_{min}}}$, or stability limit, is obtained in terms of the distribution of the kinematic viscosity and the product of the mean density $\overline{\rho^*}$ and mean vorticity $\frac{d\overline{u^*}}{dy^*}$ across the boundary layer. With the help of this estimate for $R_{\theta_{cr_{min}}}$, it is shown that withdrawing heat from the fluid through the solid surface increases $R_{\theta_{cr_{min}}}$ and stabilizes the flow, as compared with the flow over an insulated surface at the same Mach number. Conduction of heat to the fluid through the solid surface has exactly the opposite effect. The value of $R_{\theta_{cr_{min}}}$ for the insulated surface decreases as the Mach number increases for the case of a uniform free-stream velocity. These general conclusions are supplemented by detailed calculations of the curves of wave number (inverse wave length) against Reynolds number for the neutral disturbances for 10 representative cases of insulated and noninsulated surfaces.

So far as laminar stability is concerned, an important difference exists between the case of a subsonic and supersonic free-stream velocity outside the boundary layer. The neutral boundary-layer disturbances that are significant for laminar stability die out exponentially with distance from the solid surface; therefore the phase velocity c^* of these disturbances is subsonic relative to the free-stream velocity $\overline{u_o^*}$ - or $\overline{u_o^*} - c^* < \overline{a_o^*}$, where $\overline{a_o^*}$ is the local sonic velocity. When $\frac{\overline{u_o^*}}{\overline{a_o^*}} = M_o < 1$ (where M_o is free-stream Mach number), it follows that $0 \leqq c^* \leqq c^*_{max}$; and any

laminar boundary-layer flow is ultimately unstable at sufficiently high Reynolds numbers because of the destabilizing action of viscosity near the solid surface, as explained by Prandtl for the incompressible fluid. When $M_o > 1$, however, $\frac{c^*}{u_o^*} > 1 - \frac{1}{M_o} > 0$. If the quantity $\left[\frac{d}{dy^*}\left(\overline{\rho^*}\,\frac{d\overline{u^*}}{dy^*}\right)\right]_{\overline{u^*}=\overline{c^*}}$ is large enough negatively, the rate at which energy passes from the disturbance to the mean flow, which is proportional to $-c^*\left[\frac{d}{dy^*}\left(\overline{\rho^*}\,\frac{d\overline{u^*}}{dy^*}\right)\right]_{\overline{u^*}=c^*}$, can always be large enough to counterbalance the rate at which energy passes from the mean flow to the disturbance because of the destabilizing action of viscosity near the solid surface. In that case only damped disturbances exist and the laminar boundary layer is completely stable at all Reynolds numbers. This condition occurs when the rate at which heat is withdrawn from the fluid through the solid surface reaches or exceeds a critical value that depends only on the Mach number and the properties of the gas. Calculations show that for $M_o > 3$ (approx.) the laminar boundary-layer flow for thermal equilibrium - where the heat conduction through the solid surface balances the heat radiated from the surface - is completely stable at all Reynolds numbers under free-flight conditions if the free-stream velocity is uniform.

The results of the analysis of the stability of the laminar boundary layer must be applied with care to discussions of transition; however, withdrawing heat from the fluid through the solid surface, for example, not only increases $R_{\theta_{cr_{min}}}$ but also decreases the initial rate of amplification of the self-excited disturbances, which is roughly proportional to $1/\sqrt{R_{\theta_{cr_{min}}}}$. Thus, the effect of the thermal conditions at the solid surface on the transition Reynolds number $R_{\theta_{tr}}$ is similar to the effect on $R_{\theta_{cr_{min}}}$. A comparison between this conclusion and experimental investigations of the effect of surface heating on transition at low speeds shows that the results of the present paper give the proper direction of this effect.

The extension of the results of the stability analysis to laminar boundary-layer gas flows with a pressure gradient in the direction of the free stream is discussed.

INTRODUCTION

By the theoretical studies of Heisenberg, Tollmien, Schlichting, and Lin (references 1 to 5) and the careful experimental investigations of Liepmann (reference 6) and H. L. Dryden and his associates (reference 7), it has been definitely established that the flow in the laminar boundary layer of a viscous homogeneous incompressible fluid is unstable above a certain characteristic critical Reynolds number. When the level of the disturbances in the free stream is low, as in most cases of technical interest, this inherent instability of the laminar motion at sufficiently high Reynolds numbers is responsible for the ultimate transition to turbulent flow in the boundary layer. The steady laminar boundary-layer flow would always represent a possible solution of the steady equations of motion, but this steady flow is in a state of unstable dynamic equilibrium above the critical Reynolds number. Self-excited disturbances (Tollmien waves) appear in the flow, and these disturbances grow large enough eventually to destroy the laminar motion.

The question naturally arises as to how the phenomena of laminar instability and transition to turbulent flow are modified when the fluid velocities and temperature variations in the boundary layer are large enough so that the compressibility and conductivity of the fluid can no longer be neglected. The present paper represents the second phase of a theoretical investigation of the stability of the laminar boundary-layer flow of a gas, in which the compressibility and heat conductivity of the gas as well as its viscosity, are taken into account. The first part of this work was presented in reference 8. The objects of this investigation are (1) to determine how the stability of the laminar boundary layer is affected by the free-stream Mach number and the thermal conditions at the solid boundary and (2) to obtain a better understanding of the physical basis for the instability of laminar gas flows. In this sense, the present study is an extension of the Tollmien-Schlichting analysis of the stability of the laminar flow of an incompressible fluid, but the investigation is also concerned with the general question of boundary-layer disturbances in a compressible fluid and their possible interactions with the main external flow.

SYMBOLS

With minor exceptions the symbols used in this paper are the same as those introduced in reference 8. Physical quantities are denoted by an asterisk, or star, whereas the corresponding nondimensional quantities are unstarred. A bar over a quantity denotes mean value; a prime denotes a fluctuation; the subscript o denotes free-stream values at the "edge" of the boundary layer; the subscript 1 denotes values at the solid surface; and the subscript c denotes values at the inner "critical layer", where the phase velocity of the disturbance equals the mean flow velocity. The free-stream values are the characteristic measures for all nondimensional quantities. The characteristic length measure is the boundary-layer thickness δ, except where otherwise indicated. Note that in order to conform with standard notation, the symbol δ for boundary-layer thickness is unstarred, whereas the symbols δ^* and θ are used for boundary-layer displacement thickness and boundary-layer momentum thickness, respectively.

x^*	distance along surface
y^*	distance normal to surface
t^*	time
u^*	component of velocity in x^*-direction
$w = \dfrac{\overline{u^*}}{\overline{u_o^*}}$	
v^*	component of velocity in y^*-direction
$\varphi = \dfrac{v^{*\prime}}{\overline{u_o^*}}$	
ψ^*	stream function for mean flow
ρ^*	density of gas
p^*	pressure of gas
T^*	temperature of gas
τ^*	laminar shear stress
μ_1^*	ordinary coefficient of viscosity of gas
ν^*	kinematic viscosity of gas $\left(\mu_1^*/\rho^*\right)$

k^* thermal conductivity of gas

c_v specific heat at constant volume

c_p specific heat at constant pressure

R^* gas constant per gram

γ ratio of specific heats $\left(c_p/c_v\right)$; 1.405 for air

c^* complex phase velocity of boundary-layer disturbance

λ^* wave length of boundary-layer disturbance

δ boundary-layer thickness

δ^* boundary-layer displacement thickness $\left(\int_0^\infty (1 - \rho w)dy^*\right)$

θ boundary-layer momentum thickness $\left(\int_0^\infty \rho w(1 - w)dy^*\right)$

α^* wave number of boundary-layer disturbance $(2\pi/\lambda^*)$

$\alpha = \dfrac{2\pi}{\lambda^*/\delta}$

$\alpha_\theta = \dfrac{2\pi}{\lambda^*/\theta}$

R Reynolds number $\left(\dfrac{\overline{\rho_o^*}\ \overline{u_o^*}\ \delta}{\overline{\mu_{1_o}^*}}\right)$

$R_\theta = \dfrac{\overline{\rho_o^*}\ \overline{u_o^*}\ \theta}{\overline{\mu_{1_o}^*}}$

M_o Mach number $\left(\dfrac{\overline{u_o^*}}{\sqrt{\gamma R^*\ \overline{T_o^*}}}\right)$

σ Prandtl number $\left(c_p \frac{\overline{\mu_{1_0}^*}}{\overline{k_0^*}}\right)$

1. PRELIMINARY CONSIDERATIONS

In the first phase of this investigation (reference 8) the stability of the laminar boundary-layer flow of a gas is analyzed by the method of small perturbations, which was already so successfully utilized for the study of the stability of the laminar flow of an incompressible fluid. (See reference 5.) By this method a nonsteady gas flow is investigated in which all physical quantities differ from their values in a given steady gas flow by small perturbations that are functions of the time and the space coordinates. This nonsteady flow must satisfy the complete gas-dynamic equations of motion and the same boundary conditions as the given steady flow. The question is whether the nonsteady flow damps to the steady flow, oscillates about it, or diverges from it with time - that is, whether the small perturbations are damped, neutral, or self-excited disturbances in time, and thus whether the given steady gas flow is stable or unstable. The analysis is particularly concerned with the conditions for the existence of neutral disturbances, which mark the transition from stable to unstable flow and define the minimum critical Reynolds number.

In order to bring out some of the principal features of the stability problem without becoming involved in hopeless mathematical complications, the solid boundary is taken as two dimensional and of negligible curvature and the boundary-layer flow is regarded as plane and essentially parallel; that is, the velocity component in the direction normal to the surface is negligible and the velocity component parallel to the surface is a function mainly of the distance normal to the surface. The small disturbances, which are also two dimensional, are analyzed into Fourier components, or normal modes, periodic in the direction of the free stream; and the amplitude of each one of these partial oscillations is a function of the distance normal to the solid surface, that is, $u^{*\prime} = \overline{u_0^*}\, f(y)\, e^{i\alpha(x-ct)}$.

In the study of the stability of the laminar boundary layer, it will be seen that only the local properties of the "parallel" flow are significant. To include the variation of the mean velocity in the direction of the free stream or the velocity component normal to the solid boundary in the problem would lead only to higher order terms in the differential equations governing the disturbances, since both of these factors are inversely proportional to the local Reynolds number based on the boundary-layer thickness. (See, for example, reference 2.) By a careful analysis, Pretsch has shown that even with a pressure gradient in the direction of the free stream the local mean-velocity distribution alone determines the stability characteristics of the local boundary-layer flow at large Reynolds numbers (reference 9). Such a statement applies only to the stability of the flow within the boundary layer. For the interaction between the boundary layer and a main "external" supersonic flow, for example, it is obviously the variation in boundary-layer thickness and mean velocity along the surface that is significant. (See reference 10.)

The aforementioned considerations also lead quite naturally to the study of individual partial oscillations of the form $f(y)\ e^{i\alpha(x-ct)}$, for which the differential equations of disturbance do not contain x and t explicitly. Those partial oscillations are ideally suited for the study of instability, for in order to show that a flow is unstable it is unnecessary to consider the most general possible disturbance; in fact, the simplest will suffice. It is only necessary to show that a particular disturbance satisfying the equations of motion and the boundary conditions is self-excited or, in this case, that the imaginary part of the complex phase velocity c is positive.

In reference 8 the differential equations governing one normal mode of the disturbances in the laminar boundary layer of a gas were derived and studied very thoroughly. The complete set of solutions of the disturbance equations was obtained and the physical boundary conditions that these solutions satisfy were investigated. It was found that the final relation between the values of c, α, and R that determines the possible neutral disturbances (limits of stability) is of the same form in the compressible fluid as in the incompressible fluid, to a first approximation. The basis for this result is the fact that for Reynolds numbers of the order of those encountered in most aerodynamic problems, the temperature disturbances have only a negligible effect on those particular velocity solutions of the disturbance equations that depend primarily on the viscosity (viscous solutions). To a first approximation, these viscous solutions therefore do not depend directly on the heat conductivity and are of the same form as in the incompressible fluid, except that they involve the Reynolds number based on the kinematic viscosity near the solid boundary (where the viscous forces are important) rather than in the free stream. In this first approximation, the second viscosity coefficient, which is a measure of the dependence of the pressure on the rate of change of density, does not affect the stability of the laminar boundary layer. From these results it was inferred that at large Reynolds numbers the influence of the viscous forces on the stability is essentially the same as in an incompressible fluid. This inference is borne out by the results of the present paper.

The influence of the inertial forces on the stability of the laminar boundary layer is reflected in the behavior of the asymptotic inviscid solutions of the disturbance equations, which are independent of Reynolds number in first approximation. The results obtained in reference 8 show that the behavior of the inertial forces is dominated by the distribution of the product of the mean density and mean vorticity $\rho\frac{dw}{dy}$ across the boundary layer. (The gradient of this quantity, or $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$, which plays the same role as the gradient of the vorticity in the case of an incompressible fluid, is a measure of the rate at which the x-momentum of the thin layer of fluid near the critical layer (where $w = c$) increases, or decreases, because of the transport of momentum by the disturbance.) In order to clarify the behavior of the inertial forces, the limiting case of an inviscid fluid $(R \rightarrow \infty)$ is studied in detail in reference 8. The following general criterions are obtained: (1) If the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ vanishes for some value of $w > 1 - \frac{1}{M_o}$, then neutral and self-excited subsonic disturbances exist and the inviscid compressible flow is unstable. (2) If the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ does not vanish for some value of $w > 1 - \frac{1}{M_o}$, then all subsonic disturbances of finite wave length are damped and the inviscid compressible flow is stable. (Outside the boundary layer, the relative velocity between the mean flow and the x-component of the phase velocity of a subsonic disturbance is less than the mean sonic velocity. The magnitude of such a disturbance dies out exponentially with distance from the solid surface.) (3) In general, a disturbance gains energy from the mean flow if $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ is positive at the critical layer (where $w = c$) and loses energy to the mean flow if $\left[\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)\right]_{w=c} < 0$.

The general stability criterions for inviscid compressible flow give some insight into the effect of the inertial forces on the stability, but they cannot be taken over bodily to the real compressible fluid. Of course, if a flow is unstable in the limiting case of an infinite Reynolds number, the flow is unstable for a certain finite range of Reynolds number. A compressible flow that is stable when $R \longrightarrow \infty$, however, is not necessarily stable at all finite Reynolds numbers when the effect of viscosity is taken into account. One of the objects of the present paper is to settle this question.

On the basis of the stability criterions obtained in reference 8, some general statements were made concerning the effect of thermal conditions at the solid boundary on the stability of laminar boundary-layer flow. It is concluded from physical reasoning and a study of the equations of mean motion that the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ vanishes for some value of $w > 0$ if $\left(\frac{\partial T}{\partial y}\right)_1 \leqq 0$, that is, if heat is <u>added</u> to the fluid through the solid surface or if the surface is insulated. If $\left(\frac{\partial T}{\partial y}\right)_1 > 0$ and is sufficiently large, that is, if heat is <u>withdrawn</u> from the fluid through the solid surface at a sufficient rate, the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ never vanishes. Thus, when $\left(\frac{\partial T}{\partial y}\right)_1 \leqq 0$, the laminar boundary-layer flow is destabilized by the action of the inertial forces but stabilized through the increase of kinematic viscosity near the solid surface. When $\left(\frac{\partial T}{\partial y}\right)_1 > 0$, the reverse is true. The question of which of these effects is predominant can be answered only by further study of the stability problem in a real compressible fluid.

In the present paper this investigation is continued along the following lines:

(1) A study is made of how the general criterions for instability in an inviscid compressible fluid are modified by the introduction of a small viscosity (stability at very large Reynolds numbers).

(2) The conditions for the existence of neutral disturbances at large Reynolds number are examined (study of asymptotic form of relation between eigen-values of c, α, and R).

(3) A relatively simple expression for the approximate value of the minimum critical Reynolds number is derived; this expression involves the local distribution of mean velocity and mean temperature across the boundary layer. This approximation will serve as a criterion from which the effect of the free-stream Mach number and thermal conditions at the solid surface on the stability of laminar boundary-layer flow is readily evaluated. The question of the relative influence of the kinematic viscosity and the distribution of $\rho\frac{dw}{dy}$ on stability would then be settled.

(4) The energy balance for small disturbances in the real compressible fluid is considered in an attempt to clarify the physical basis for the instability of laminar gas flows.

(5) In order to supplement the investigations outlined in the four preceding paragraphs, detailed calculations are made of the limits of stability, or the curve of α against R for the neutral disturbances for several representative cases of insulated and noninsulated surfaces. The results of the calculations are presented in figures 1 to 8 and tables I to IV. The method of computation of the stability limits is briefly outlined in reference 8, although the calculations were not carried out in that paper.

In the present investigation the work of Heisenberg (reference 1) and Lin (reference 5) on the stability of a real incompressible fluid is naturally an indispensable guide. In fact, the methods utilized in the present study are analogous to those developed for an incompressible fluid.

The present paper is concerned only with the subsonic disturbances. The amplitude of the subsonic disturbance dies out rapidly with distance from the solid boundary. In other words, the neutral subsonic disturbance is an "eigen-oscillation" confined mainly to the boundary layer and exists only for discrete eigen-values of c, α, and R that determine the limits of stability of laminar boundary-layer flow. Disturbances classified in reference 8 as neutral "supersonic," that is, disturbances such that the relative velocity between the x-component of the phase velocity of such a disturbance and the free-stream velocity is greater than the local mean sound speed in the free stream, are actually progressive sound waves that impinge obliquely on the boundary layer and are reflected with change of amplitude. For disturbances of this type the wave length and phase velocity are obviously completely arbitrary (eigenvalues are continuous), and these disturbances have no significance for boundary-layer stability.

When the free-stream velocity is supersonic $\left(M_o > 1\right)$, the subsonic boundary-layer disturbances must satisfy the requirement that $\overline{u_o^*} - c^* < \overline{a_o^*}$ or $c > 1 - \frac{1}{M_o}$ (for $M_o < 1$, $c \geq 0$). Now, by analogy with the case of an incompressible fluid it is to be expected that for values of c greater than some critical value of c_o, say, all subsonic disturbances are damped. Thus, when $M_o > 1$, there is the possibility that for certain mean velocity-temperature distributions across the boundary layer, neutral or self-excited disturbances satisfying the differential equations of motion, the boundary conditions, and, also, the physical requirement that $c > 1 - \frac{1}{M_o}$ cannot be found. In that event, the laminar boundary flow is stable at all Reynolds numbers. This interesting possibility is investigated in the present paper.

2. CALCULATION OF THE LIMITS OF STABILITY OF THE LAMINAR BOUNDARY LAYER IN A VISCOUS CONDUCTIVE GAS

In order that the complete system of solutions of the differential equations for the propagation of small disturbances in the laminar boundary layer shall satisfy the physical boundary conditions, the phase velocity must depend on the wave length, the Reynolds number, and the Mach number in a manner that is determined entirely by the local distribution of mean velocity and mean temperature across the boundary layer. In other words, the only possible subsonic disturbances in the laminar boundary layer are those for which there exists a definite relation of the form (reference 8)

$$c = c\left(\alpha, R, M_o^2\right) \tag{2.1}$$

Since α, R, and M_o^2 are real quantities, the relation (2.1) is equivalent to the two relations

$$c_r = c_r\left(\alpha,\ R,\ M_0^{\,2}\right) \tag{2.1a}$$

$$c_i = c_i\left(\alpha,\ R,\ M_0^{\,2}\right) \tag{2.1b}$$

The curve $c_i\left(\alpha,\ R,\ M_0^{\,2}\right) = 0$ $\left(\text{or}\ \ \alpha = \alpha\left(R,\ M_0^{\,2}\right)\right)$ for the neutral disturbances gives the limits of stability of the laminar boundary layer at a given value of the Mach number. From this curve can be determined the value of the Reynolds number below which disturbances of all wave lengths are damped and above which self-excited disturbances of certain wave lengths appear in a given laminar boundary-layer flow.

In reference 8, it is shown that the relation (2.1) between the phase velocity and the wave length takes the following form:

$$E\left(\alpha,\ c,\ M_0^{\,2}\right) = F(z) \tag{2.2}$$

In equation (2.2), $F(z)$ is the Tietjens function (reference 11) defined by the relation

$$F(z) = 1 + \frac{\displaystyle\int_{\infty}^{-z} \zeta^{3/2} H_{1/3}^{(1)}\left\{\frac{2}{3}(i\zeta)^{3/2}\right\} d\zeta}{\displaystyle z\int_{\infty}^{-z} \zeta^{1/2} H_{1/3}^{(1)}\left\{\frac{2}{3}(i\zeta)^{3/2}\right\} d\zeta} \tag{2.3}$$

where

$$z = \left(\frac{\alpha R w_c'}{\nu_c}\right)^{1/3}\left(y_c - y_1\right) \tag{2.4}$$

and the quantity $H_{1/3}^{(1)}$ is the Hankel function of the first kind of order $1/3$. The prime denotes differentiation with respect to y. The function $E\left(\alpha,\ c,\ M_0^{\,2}\right)$, which depends only on the asymptotic inviscid solutions φ_1 and φ_2 (section 4 of reference 8) and not on the Reynolds number, is defined as follows:

$$\left(y_1 - y_c\right) E\left(\alpha, c, M_o^2\right) = \frac{\begin{vmatrix} \varphi_{11} & \varphi_{12}' + \beta\varphi_{12} \\ \varphi_{21} & \varphi_{22}' + \beta\varphi_{22} \end{vmatrix}}{\begin{vmatrix} \dfrac{T_1\varphi_{11}' + M_o^2 w_1' c\varphi_{11}}{T_1 - M_o^2 c^2} & \varphi_{12}' + \beta\varphi_{12} \\ \dfrac{T_1\varphi_{21}' + M_o^2 w_1' c\varphi_{21}}{T_1 - M_o^2 c^2} & \varphi_{22}' + \beta\varphi_{22} \end{vmatrix}} \tag{2.5}$$

where

$$\left.\begin{aligned} \beta &= \alpha\sqrt{1 - M_o^2(1 - c)^2} \\ \varphi_{ij} &= \varphi_i\left(y_j\right) \\ i, j &= 1, 2 \end{aligned}\right\} \tag{2.6}$$

and y_1 and y_2 are the coordinates of the solid surface and the "edge" of the boundary layer, respectively.

The Tietjens function was carefully recalculated in reference 8, and the real and imaginary parts of the function $\Phi(z) = \dfrac{1}{1 - F(z)}$ are plotted in figure 9. (The function $\Phi(z)$ is found to be more suitable then $F(z)$ for the actual calculation of the stability limits.)

The inviscid solutions φ_1 and φ_2 were obtained as power series in α^2 as follows (section 8 of reference 8):

$$\varphi_1\left(y;\ \alpha^2,\ c,\ M_o^2\right) = (w - c)\sum_{n=0}^{\infty} \alpha^{2n} h_{2n}\left(y;\ c,\ M_o^2\right) \tag{2.7}$$

$$\varphi_2\left(y;\ \alpha^2,\ c,\ M_o^2\right) = (w - c)\sum_{n=0}^{\infty} \alpha^{2n} k_{2n+1}\left(y;\ c,\ M_o^2\right) \tag{2.8}$$

where for $n \geqq 1$

$$h_{2n}\left(y;\ c,\ M_o^2\right) = \int_{y_1}^{y} \left[\frac{T}{(w - c)^2} - M_o^2\right] dy \int_{y_1}^{y} \frac{(w - c)^2}{T} h_{2n-2}\left(y;\ c,\ M_o^2\right) dy \tag{2.9}$$

and

$$h_o = 1.0$$

and for $n \geqq 1$

$$k_{2n+1}\left(y;\ c, M_o^2\right) = \int_{y_1}^{y} \left[\frac{T}{(w - c)^2} - M_o^2\right] dy \int_{y_1}^{y} \frac{(w - c)^2}{T} k_{2n-1}\left(y;\ c,\ M_o^2\right) dy \tag{2.10}$$

and

$$k_1\left(y;\ c,\ M_o^2\right) = \int_{y_1}^{y} \left[\frac{T}{(w - c)^2} - M_o^2\right] dy$$

The lower limit in the integrals is taken at the surface merely for convenience. When $y > y_c$, the path of integration must be taken below the point $y = y_c$ in the complex y-plane. The power series in α^2 are then uniformly convergent for any finite value of α.

At the surface, the inviscid solutions are readily evaluated

$$\left.\begin{aligned}
\varphi_{11} &= -c \\
\varphi_{11}' &= w_1' \\
\varphi_{21} &= 0 \\
\varphi_{21}' &= -\frac{1}{c}\left(T_1 - M_o^2c^2\right)
\end{aligned}\right\} \tag{2.11}$$

At the "edge" of the boundary layer, the inviscid solutions are most conveniently expressed as follows:

$$\left.\begin{aligned}
\varphi_{12} &= (1 - c) \sum_{n=0}^{\infty} \alpha^{2n} H_{2n}\left(c, M_o^2\right) \\
\varphi_{22} &= (1 - c) \sum_{n=0}^{\infty} \alpha^{2n} K_{2n+1}\left(c, M_o^2\right) \\
\varphi_{12}' &= (1 - c) \left[\frac{1 - M_o^2(1 - c)^2}{(1 - c)^2}\right] \sum_{n=1}^{\infty} \alpha^{2n} H_{2n-1}\left(c, M_o^2\right) \\
\varphi_{22}' &= (1 - c) \left[\frac{1 - M_o^2(1 - c)^2}{(1 - c)^2}\right] \sum_{n=0}^{\infty} \alpha^{2n} K_{2n}\left(c, M_o^2\right)
\end{aligned}\right\} \tag{2.12}$$

where

$$
\left.
\begin{aligned}
H_{2n}\left(c,\ M_o^2\right) &= h_{2n}\left(y_2;\ c,\ M_o^2\right) \\
H_o &= 1.0 \\
K_{2n+1}\left(c,\ M_o^2\right) &= k_{2n+1}\left(y_2;\ c,\ M_o^2\right) \\
H_{2n-1}\left(c,\ M_o^2\right) &= \left[\frac{1 - M_o^2(1 - c)^2}{(1 - c)^2}\right]^{-1} h_{2n}'\left(y_2;\ c,\ M_o^2\right) \\
K_{2n}\left(c,\ M_o^2\right) &= \left[\frac{1 - M_o^2(1 - c)^2}{(1 - c)^2}\right]^{-1} k_{2n+1}'\left(y_2;\ c,\ M_o^2\right) \\
K_o &= 1.0
\end{aligned}
\right\} \tag{2.13}
$$

With the aid of equations (2.11), the expression for $E\left(\alpha,\ c,\ M_o^2\right)$ can be rewritten as follows:

$$
E\left(\alpha,\ c,\ M_o^2\right) = \frac{1}{1 + \lambda(c)} \frac{w_1'\left(\varphi_{22}' + \beta\varphi_{22}\right)}{w_1'\left(\varphi_{22}' + \beta\varphi_{22}\right) + \frac{T_1}{c}\left(\varphi_{12}' + \beta\varphi_{12}\right)} \tag{2.14}
$$

where

$$
\lambda(c) = \frac{w_1'\left(y_c - y_1\right)}{c} - 1 \tag{2.15}
$$

The relation (2.2) between the phase velocity and the wave length is brought into a form more suitable for the calculation of the stability limits by making use of the fact that for real values of c the imaginary part of $E\left(\alpha,\ c,\ M_o^2\right)$ is contributed largely by the integral $K_1\left(c, M_o^2\right)$. (The procedure to be followed is identical with that used by Lin in the limiting case of the incompressible fluid (reference 5, part III).) Define the function $\Phi(z)$ by the relation

$$\Phi(z) = \frac{1}{1 - F(z)} \tag{2.16}$$

Then,

$$\Phi(z) = \frac{1}{1 - E} = (1 + \lambda) \frac{(u + iv)}{1 + \lambda(u + iv)} \tag{2.17}$$

where

$$u + iv = 1 + \frac{w_1'c}{T_1}\left(\frac{\varphi_{22}' + \beta\varphi_{22}}{\varphi_{12}' + \beta\varphi_{12}}\right) \tag{2.18}$$

Equation (2.17) is equivalent to the two real relations

$$\Phi_i(z) = \frac{(1 + \lambda)v}{(1 + \lambda u)^2 + \lambda^2 v^2} \tag{2.19}$$

$$\Phi_r(z) = (1 + \lambda)\left[\frac{u(1 + \lambda u) + \lambda v^2}{(1 + \lambda u)^2 + \lambda^2 v^2}\right] \tag{2.20}$$

The real and imaginary parts of $\Phi(z)$ are plotted against z in figure 9.

The dominant term in the imaginary part of the right-hand side of equation (2.18), which involves $K_1\left(c, M_o^2\right)$, is extracted by means of straightforward algebraic transformations. Relation (2.18) becomes

$$u+iv=\frac{w_1{}'c}{T_1}\left[\left(K_1+\frac{T_1}{w_1{}'c}\right)+\left(\frac{1-\alpha^2H_2}{\alpha}\right)\frac{\dfrac{\sqrt{1-M_o{}^2(1-c)^2}}{(1-c)^2}\left(1-\sum_{n=1}^{\infty}\alpha^{2n}N_{2n}\right)-\sum_{n=1}^{\infty}\alpha^{2n+1}N_{2n+1}}{\left(1-\sum_{n=2}^{\infty}\alpha^{2n}M_{2n}\right)+\dfrac{\sqrt{1-M_o{}^2(1-c)^2}}{(1-c)^2}\left(\alpha H_1-\sum_{n=1}^{\infty}\alpha^{2n+1}M_{2n+1}\right)}\right] \tag{2.21}$$

where

$$N_2 \equiv H_2$$

and for $n \geqq 3$

$$N_n = K_1H_{n-1} - K_n \tag{2.22a}$$

and

$$M_n = E_2H_{n-2} - E_n \tag{2.22b}$$

When c is real,

$$v \approx \frac{w_1{}'c}{T_1}\,\text{I.P.}\ K_1$$

for those values of α and c that occur in the stability calculations. (This approximation is justified later in appendix A.) The imaginary part of the integral $K_1\left(c,\ M_o{}^2\right)$ is readily computed. It is found that

$$\text{I.P. } K_1\left(c, M_0^2\right) \approx -\pi \frac{T_c^2}{\left(w_c'\right)^3}\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c}$$

$$= -\pi \frac{T_c}{\left(w_c'\right)^2}\left(\frac{w_c''}{w_c'} - \frac{T_c'}{T_c}\right) \qquad (2.23)$$

Now $\lambda(c)$ is generally quite small, therefore $\Phi_i(z)$ can be taken equal to $v(c)$ and $\Phi_r(z)$ can be taken equal to u as a zeroth approximation. From equations (2.19) and (2.20), when c is real

$$\Phi_i^{(0)}\left(z^{(0)}\right) = v = -\frac{\pi w_1' c}{T_1} \frac{T_c}{\left(w_c'\right)^2}\left(\frac{w_c''}{w_c'} - \frac{T_c'}{T_c}\right) \qquad (2.24)$$

$$u^{(0)} = \Phi_r^{(0)}\left(z^{(0)}\right) \qquad (2.25)$$

By equation (2.24), $z^{(0)}$ is related to c with the aid of figure 9; and by equation (2.25), $u^{(0)}$ is also related to c. The quantity αR is connected with c by means of the identity

$$\alpha R = \frac{v_c}{w_c'(1 + \lambda)^3}\left(\frac{z w_1'}{c}\right)^3 \qquad (2.26)$$

and the corresponding values of α are obtained from equation (2.21) (slightly transformed) by a method of successive approximations.

Thus,

$$\alpha = \frac{\dfrac{w_1{}'c}{T_1}\left(1 - \alpha^2 H_2\right)\left[\dfrac{\sqrt{1 - M_o{}^2(1 - c)^2}}{(1 - c)^2}\left(1 - \sum_{n=1}^{\infty} \alpha^{2n} N_{2n}\right) - \sum_{n=1}^{\infty} \alpha^{2n+1} N_{2n+1}\right]}{(u - L)\left[1 - \sum_{n=2}^{\infty} \alpha^{2n} M_{2n} + \dfrac{\sqrt{1 - M_o{}^2(1 - c)^2}}{(1 - c)^2}\left(\alpha H_1 - \sum_{n=1}^{\infty} \alpha^{2n+1} M_{2n+1}\right)\right]} \tag{2.27}$$

where

$$L = \frac{w_1{}'c}{T_1}\,\text{R.P.}\left(K_1 + \frac{T_1}{w_1{}'c}\right)$$

(The symbols M_k and N_k now designate the real parts of the integrals M_k and N_k.) The iteration process is begun by taking a suitable initial value of α on the right-hand side of equation (2.27). The methods adopted for computing these integrals when the mean velocity-temperature profile is known are described in appendixes A to C.

For greater accuracy, the values of z and u for a given real value of c are computed by successive approximations. From equations (2.19) and (2.20),

$$\phi_i{}^{(n+1)}\left(z^{(n+1)}\right) = \frac{(1 + \lambda)v}{\left(1 + \lambda u^{(n)}\right)^2 + \lambda^2 v^2} \tag{2.28}$$

$$u^{(n+1)} = \phi_r{}^{(n+1)}\left(z^{(n+1)}\right)\left[\frac{\left(1 + \lambda u^{(n)}\right)^2 + \lambda^2 v^2}{(1 + \lambda)\left(1 + \lambda u^{(n)}\right)}\right] - \frac{\lambda v^2}{1 + \lambda u^{(n)}} \tag{2.29}$$

The value of v is always approximated by relation (2.24).

Curves of wave number against Reynolds number for the neutral disturbance have been calculated for 10 representative cases (fig. 4), that is, insulated surface at Mach numbers of 0, 0.50, 0.70, 0.90, 1.10, and 1.30 and heat transfer across the solid surface at a Mach number of 0.70 with values of the ratio of surface temperature to free-stream temperature T_1 of 0.70, 0.80, 0.90, and 1.25. (It is found more desirable to base the nondimensional wave number and the Reynolds number on the momentum thickness θ, which is a direct measure of the skin friction, rather than on the boundary-layer thickness δ, which is somewhat indefinite.)

In figure 5 the minimum critical Reynolds number $R_{\theta_{cr_{min}}}$, or the stability limit, is plotted against Mach number for the insulated surface; and in figure 6(a) $R_{\theta_{cr_{min}}}$ is plotted against T_1 for the cooled or heated surface at a Mach number of 0.70. The marked stabilizing influence of a withdrawal of heat from the fluid is clearly evident. Discussion of the physical significance of these numerical results is reserved until after general criterions for the stability of the laminar boundary layer have been obtained.

3. DESTABILIZING INFLUENCE OF VISCOSITY AT VERY LARGE REYNOLDS NUMBERS; EXTENSION OF HEISENBERG'S CRITERION TO THE COMPRESSIBLE FLUID

The numerical calculation of the limits of stability for several particular cases gives some indication of the effects of free-stream Mach number and thermal conditions at the solid surface on the stability of the laminar boundary layer. It would be very desirable, however, to establish general criterions for laminar instability. For the incompressible fluid, Heisenberg has shown that the influence of viscosity is generally destabilizing at very large Reynolds numbers (reference 1). His criterion can be stated as follows: If a neutral disturbance of nonvanishing phase velocity and finite wave length exists in an inviscid fluid ($R \rightarrow \infty$) for a given mean velocity distribution, a disturbance of the same wave length is unstable, or self-excited, in the real fluid at very large (but finite) Reynolds numbers.

The same conclusion can be drawn from Prandtl's discussion of the energy balance for small disturbances in the laminar boundary layer (reference 12).

Heisenberg's criterion is established for subsonic disturbances in the laminar boundary layer of a compressible fluid by an argument quite similar to that which he gave originally for the incompressible fluid and which was later supplemented by Lin (reference 5, part III).

At very large Reynolds numbers, the relation (2.1) between the phase velocity and the wave length can be considerably simplified. When λ is finite and c does not vanish, $|z| \gg 1$ at large Reynolds numbers. The asymptotic behavior of the Tietjens function $F(z)$ as $|z| \longrightarrow \infty$ is given by (reference 5, part I)

$$\left(y_1 - y_c\right) F(z) = \frac{-e^{\pi i/4}}{\sqrt{\alpha \frac{R}{\upsilon_c} c}} \tag{3.1}$$

and the relation (2.1) becomes

$$\left(y_1 - y_c\right) E\left(\alpha,\ c,\ M_o^2\right) = E_1\left(\alpha,\ c,\ M_o^2\right) = \frac{-e^{\pi i/4}}{\sqrt{\alpha \frac{R}{\upsilon_c} c}} \tag{3.2}$$

where $E\left(\alpha,\ c,\ M_o^2\right)$ is given by equation (2.14).

Suppose that a neutral disturbance of nonvanishing wave number $\alpha_s = \frac{2\pi}{\lambda_s}$ and phase velocity $c_s > 1 - \frac{1}{M_o}$ exists in the inviscid fluid (limiting case of an infinite Reynolds number). The phase velocity c is a continuous function of R, and for a disturbance of given wave number α_s the value of c at very large Reynolds numbers will differ from c_s by a small increment Δc. Both sides of equation (3.2) can be developed in a Taylor's series in Δc, and an expression for Δc can be obtained as follows:

$$E_1\left(\alpha,\ c,\ M_o^2\right) = E_1\left(\alpha_s,\ c_s,\ M_o^2\right) + \left(\frac{\partial E_1}{\partial c}\right)_{c_s, \alpha_s} \Delta c + \ldots$$

$$= \frac{-e^{\pi i/4}}{\sqrt{\alpha_s \frac{R}{\upsilon_{c_s}} c_s}} \left[1 + O(\Delta c)\right] \tag{3.3}$$

The boundary condition

$$\varphi_{22}'\left(\alpha_s,\ c_s,\ M_o^2\right) + \beta_s\varphi_{22}\left(\alpha_s,\ c_s,\ M_o^2\right) = 0 \tag{3.4}$$

must be satisfied for the inviscid neutral disturbance, and the function $E_1\left(\alpha_s,\ c_s,\ M_o^2\right)$ vanishes (equation 2.14). Recognizing that

$$\left(\frac{\partial E_1}{\partial c}\right)_{c_s,\alpha_s} \gg \frac{1}{\sqrt{\alpha_s \dfrac{R}{\upsilon_{c_s}} c_s}}$$

reduces equation (3.3) for Δc to the form

$$\Delta c = \frac{-e^{\pi i/4}}{\sqrt{\alpha_s \dfrac{R}{\upsilon_{c_s}} c_s \left(\dfrac{\partial E_1}{\partial c}\right)_{c_s,\alpha_s}}} \tag{3.5}$$

From equation (2.14),

$$\left(\frac{\partial E_1}{\partial c}\right)_{c_s,\alpha_s} = -\frac{c_s{}^2}{T_1}\,\frac{\left\{\dfrac{\partial}{\partial c}\left[\varphi_{22}'\left(\alpha_s,\ c,\ M_o^2\right) + \beta\varphi_{22}\left(\alpha_s,\ c,\ M_o^2\right)\right]\right\}_{c=c_s}}{\varphi_{12}'\left(\alpha_s,\ c_s,\ M_o^2\right) + \beta_s\varphi_{12}\left(\alpha_s,\ c_s,\ M_o^2\right)} \tag{3.6}$$

By equations (2.12) and the boundary condition (3.4), the quantity $\left(\frac{\partial E_1}{\partial c}\right)_{c_s,\alpha_s}$ is evaluated as follows:

$$\left(\frac{\partial E_1}{\partial c}\right)_{c_s,\alpha_s} = -\frac{c_s^2}{T_1}\,\frac{(1-c_s)^2\sum_{n=1}^{\infty}\alpha_s^{2n-1}K_{2n-1}'(c_s,M_o^2)+\sqrt{1-M_o^2(1-c_s)^2}\sum_{n=1}^{\infty}\alpha_s^{2n}K_{2n}'(c_s,M_o^2)+\frac{2-M_o^2(1-c_s)^2}{\sqrt{1-M_o^2(1-c_s)^2}}\,\frac{1}{(1-c_s)}\sum_{n=0}^{\infty}\alpha_s^{2n}K_{2n}(c_s,M_o^2)}{(1-c_s)^2\sum_{n=0}^{\infty}\alpha_s^{2n+1}H_{2n}(c_s,M_o^2)+\sqrt{1-M_o^2(1-c_s)^2}\sum_{n=1}^{\infty}\alpha_s^{2n}H_{2n-1}(c_s,M_o^2)} \tag{3.7}$$

where the primes now denote differentiation with respect to c. For small values of c_s and α_s, the quantity $\left(\frac{\partial E_1}{\partial c}\right)_{c_s,\alpha_s}$ is given approximately by the relation

$$\left(\frac{\partial E_1}{\partial c}\right)_{c_s,\alpha_s} = -\frac{c_s^2}{T_1}\left[\frac{2 - M_o^2(1 - c_s)^2}{\alpha_s(1 - c_s)^3\sqrt{1 - M_o^2(1 - c_s)^2}} + K_1'(c_s,\ M_o^2)\right] \tag{3.8}$$

and the expression for Δc is

$$\Delta c = \frac{T_1}{\sqrt{\alpha_s \frac{R}{\upsilon_{c_s}} c_s^5}} \frac{\alpha_s \sqrt{1 - M_o^2(1 - c_s)^2}\, e^{\pi i/4}}{\frac{2 - M_o^2(1 - c_s)^2}{(1 - c_s)^3} + \alpha_s \sqrt{1 - M_o^2(1 - c_s)^2}\, K_1'(c_s, M_o^2)} \tag{3.9}$$

Evaluation of the integral $K_1(c, M_o^2)$ yields the following result:

$$K_1(c, M_o^2) = -\frac{T_1}{w_1'c} + \frac{T_c^2}{(w_c')^3}\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c}(\ln c - i\pi) + O(1) \tag{3.10}$$

Since the quantity $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c_s}$ vanishes (reference 8), differentiation of equation (3.10) gives

$$K_1'(c_s, M_o^2) = \frac{T_1}{w_1'c_s^2} + \left(\frac{\partial}{\partial c}\left\{\frac{T_c^2}{(w_c')^3}\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c}\right\}\right)_{c=c_s}(\ln c_s - i\pi) + O(1) \tag{3.11}$$

Thus, $K_1'(c_s, M_o^2)$ is approximately real and positive for small values of c_s. With $c_s > 1 - \frac{1}{M_o}$, <u>I.P. Δc must also be positive</u> (equation (3.9)); therefore, a subsonic disturbance of wave length $\lambda_s \neq 0$, which is neutral in the inviscid compressible fluid, is self-excited in the real compressible fluid at very large (but finite) Reynolds numbers.

In reference 8, it was proved that a neutral subsonic boundary-layer disturbance of nonvanishing phase velocity and finite wave length exists in an inviscid compressible fluid only if the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ vanishes for some value of $w > 1 - \frac{1}{M_o}$. If this condition is satisfied, then self-excited subsonic disturbances also exist in the fluid, and the laminar boundary layer is unstable in the limiting case of an infinite Reynolds number. By the extension of Heisenberg's criterion to the compressible fluid, it can be seen that, far from stabilizing the flow, the small viscosity in the real fluid has, on the contrary, a destabilizing influence at very large Reynolds numbers. Thus, any laminar boundary-layer flow in a viscous conductive gas for which the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ vanishes for some value of $w > 1 - \frac{1}{M_o}$ is unstable at sufficiently high (but finite) Reynolds numbers.

Unless the condition $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right) = 0$ for some value of $w > 1 - \frac{1}{M_o}$ is satisfied, all subsonic disturbances of finite wave length are damped in the limiting case of infinite Reynolds number, and the inviscid flow is stable. Since the effect of viscosity is destabilizing at very large Reynolds numbers, however, a laminar boundary flow that is stable in the limit of infinite Reynolds number is not necessarily stable at large Reynolds numbers when the viscosity of the fluid is considered. (See fig. 4(l).) In fact, for the incompressible fluid, Lin has shown that every laminar boundary-layer flow is unstable at sufficiently high Reynolds numbers, whether or not the vorticity gradient $\frac{d^2w}{dy^2}$ vanishes (reference 5, part III). In order to settle this question for the compressible fluid in general terms, the relation (2.1) between the complex phase velocity and the wave length at large Reynolds numbers must now be studied for flows in which the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ does not vanish for any value of $w > 1 - \frac{1}{M_o}$.

4. STABILITY OF LAMINAR BOUNDARY LAYER AT LARGE REYNOLDS NUMBERS

The neutral subsonic disturbance marks a possible "boundary" between the damped and the self-excited disturbance, that is, between stable and unstable flow. Thus, the general conditions under which self-excited disturbances exist in the laminar boundary layer at large Reynolds numbers can be determined from a study of the behavior of the curve of α against R for the neutral disturbances. When the mean free-stream velocity is subsonic $\left(M_o < 1\right)$, the physical situation for the subsonic disturbances at large Reynolds numbers is quite similar to the analogous situation for the incompressible fluid. The curve of α against R for the neutral disturbances can be expected to have two distinct asymptotic branches that enclose a region of instability in the α,R-plane, regardless of the local distribution of mean velocity and mean temperature across the boundary layer. When the mean free-stream velocity is supersonic $\left(M_o > 1\right)$ the situation is somewhat different; under certain conditions (soon to be defined) a neutral or a self-excited subsonic disturbance $\left(c > 1 - \frac{1}{M_o}\right)$ cannot exist at any value of the Reynolds number. For this reason, it is more convenient to study the case of subsonic and supersonic free-stream velocity separately.

a. Subsonic Free-Stream Velocity $\left(M_o < 1\right)$

The asymptotic behavior at large Reynolds numbers of the curve of α against R for the neutral disturbances is determined by the relations (2.19) to (2.22) between α, R, and c for real values of c. For small values of α and c, these relations are given approximately by

$$v(c) = \Phi_i(z) = -\frac{\pi w_1{'}c}{T_1}\frac{T_c^2}{\left(w_c{'}\right)^3}\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c} \tag{4.1}$$

$$u = \Phi_r(z) \tag{4.2}$$

$$R = \frac{T_1^{1.76}}{\alpha}\left(\frac{z}{c}\right)^3 \left(w_1'\right)^2 \tag{4.3}$$

$$\alpha = \frac{w_1'c}{T_1}\frac{1}{u}\sqrt{1 - M_o^2(1 - c)^2} \tag{4.4}$$

As $R \longrightarrow \infty$, either $z \longrightarrow \infty$ or z remains finite while both α and c approach 0. These two possibilities correspond to two asymptotic branches of the curve of α against R.

Lower branch.- If z remains finite as $R \longrightarrow \infty$, then $c \longrightarrow 0$; and by equation (4.1), $\Phi_i(z) \longrightarrow 0$. Therefore, $z \longrightarrow 2.29$ while $u \longrightarrow 2.29$ (fig. 9). From equations (4.3) and (4.4), along the lower branch of the curve of α against R for neutral stability

$$R \approx \frac{\left(w_1'\right)^5\left(1 - M_o^2\right)^{3/2}}{T_1^{1.24}}\frac{1}{\alpha^4} \tag{4.5}$$

$$c \approx 2.29\frac{T_1}{w_1'\sqrt{1 - M_o^2}}\alpha \tag{4.6}$$

and $\alpha \longrightarrow 0$ at large Reynolds numbers (fig. 4(l)).

Upper branch.- Along the upper branch of the curve of α against R for neutral stability, $z \longrightarrow \infty$ and

$$\Phi_i(z) = -\frac{\pi w_1'c}{T_1}\frac{T_c^2}{\left(w_c'\right)^3}\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c} \longrightarrow \frac{1}{\sqrt{2z^3}} \approx \frac{w_1'}{\sqrt{2\alpha\frac{R}{\upsilon_c}c^3}} \tag{4.7}$$

while $u \longrightarrow 1.0$ (fig. 9 and equation (4.2)). If the quantity $\frac{d}{dy}\left(\frac{w'}{T}\right)$ does not vanish for any value of $w > 0$, then by equation (4.7) c must approach zero as $z \longrightarrow \infty$. Along this branch,

$$R \approx \frac{\left(w_1'\right)^{11}}{2\pi^2 T_1^{5.24}} \frac{\left(1 - M_o^2\right)^{5/2}}{\left\{\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_1\right\}^2} \frac{1}{\alpha^6} \tag{4.8}$$

$$c \approx \frac{T_1}{w_1'\sqrt{1 - M_o^2}} \alpha \tag{4.9}$$

and $\alpha \longrightarrow 0$ at large Reynolds numbers (fig. 4(l)).

On the other hand, if $\frac{d}{dy}\left(\frac{w'}{T}\right)$ vanishes for some value of $w \doteq c_s > 0$, then by equation (4.7), $c \longrightarrow c_s$ and $\alpha \longrightarrow \alpha_s$ as both z and R approach ∞. Now,

$$\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c} = \left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1 \frac{c - c_s}{w_1'} + \left\{\left[\frac{d^3}{dy^3}\left(\frac{w'}{T}\right)\right]_1 - \left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right] \frac{w_1''}{w_1'}\right\} \frac{c^2 - c_s^2}{2\left(w_1'\right)^2} + \ldots \tag{4.10}$$

If $\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1$ does not vanish (see appendix D), then by equations (4.4) and (4.7), along the upper branch of the curve of α against R for the neutral disturbances,

$$R \approx \frac{\left(w_1'\right)^8}{2\pi^2 T_1^{0.24}} \frac{1}{\left\{\left[\frac{d^2}{dy^2}\left(\frac{w'}{T_1}\right)\right]_1\right\}^2} \frac{1}{\alpha c^5} \frac{1}{\left(c - c_s\right)^2} \tag{4.11}$$

$$\alpha \approx \frac{w_1' c}{T_1}\sqrt{1 - M_o^2(1 - c)^2} \tag{4.12}$$

and $c \longrightarrow c_s \neq 0$, $\alpha \longrightarrow \alpha_s \neq 0$ at large Reynolds numbers (figs. 4(k) and 4(l)). If $\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1$ vanishes, the relation (4.11) is replaced by

$$R \approx \frac{2\left(w_1'\right)^{10}}{\pi^2 T_1^{0.24}} \frac{1}{\left\{\left[\frac{d^3}{dy^3}\left(\frac{w'}{T}\right)\right]_1\right\}^2} \frac{1}{\alpha c^5} \frac{1}{\left(c^2 - c_s^2\right)^2} \tag{4.13}$$

which reduces to the relation obtained by Lin in the limiting case of an incompressible fluid when $M_o \longrightarrow 0$, the solid boundary is insulated, and $w'' = 0$ for some value of $w = c_s > 0$. (See equation (12.22) of reference 5, part III.)

If the quantity $\frac{d}{dy}\left(\frac{w'}{T}\right)$ vanishes <u>at</u> the solid boundary (that is, for $w = 0$), it can be shown from the equations of motion (appendix D) that $\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1$ is always positive - except in the limiting case of an incompressible fluid. For small values of y, the quantities $\frac{d}{dy}\left(\frac{w'}{T}\right)$ and $\frac{w'}{T}$ are both positive and increasing.

For large values of y, however, $\frac{w'}{T} \longrightarrow 0$, physically; therefore $\frac{w'}{T}$ must have a maximum, or $\frac{d}{dy}\left(\frac{w'}{T}\right) = 0$ for some value of $w > 0$, and this case is no different from the general case treated in the preceding paragraph. In the limiting case of an incompressible fluid, when w'' vanishes at the surface, $w_c'' = w_1^{iv} \frac{c^2}{2\left(w_1'\right)^2}$ since w_1''' always vanishes in this case. From equation (4.8) the relation between α and R along the upper branch of the neutral stability curve is therefore

$$R \approx \frac{2\left(w_1'\right)^{19}}{\pi^2} \frac{1}{\left(w_1^{iv}\right)^2} \frac{1}{\alpha^{10}} \tag{4.14}$$

which is identical with equation (12.19) in reference 5, part III.

Thus, regardless of the behavior of the quantity $\frac{d}{dy}\left(\frac{w'}{T}\right)$ - regardless of the local distribution of mean velocity and mean temperature across the boundary layer - when $M_o < 1$, the curve of α against R for the neutral disturbances has two distinct branches at large Reynolds numbers. From physical considerations, all subsonic disturbances must be damped when the wave length is sufficiently small (α large) or the Reynolds number is sufficiently low. Consequently, the two branches of the curve of α against R for the neutral disturbances must join eventually, and the region between them in the α,R-plane is a region of instability; that is, at a given value of the Reynolds number, subsonic disturbances with wave lengths lying between two critical values λ_1 and λ_2 (α_1 and α_2) are self-excited. Thus, when $M_o < 1$, any laminar boundary-layer flow in a viscous conductive gas is unstable at sufficiently high (but finite) Reynolds numbers.

The lower branch of the curve of α against R for the neutral disturbances is virtually unaffected by the distribution of $\frac{d}{dy}\left(\frac{w'}{T}\right)$ across the boundary layer, but for the upper branch the behavior of the quantity $\frac{d}{dy}\left(\frac{w'}{T}\right)$ is decisive. When $\frac{d}{dy}\left(\frac{w'}{T}\right) = 0$ for some value of $w = c_s > 0$, the neutral subsonic disturbance passes continuously into the characteristic inviscid disturbance $c = c_s$ and $\alpha = \alpha_s$ as $R \longrightarrow \infty$. This result is in accordance with the results obtained in reference 9 for the inviscid compressible fluid and is in agreement with Heisenberg's criterion. In addition, all subsonic disturbances of finite wave length $\lambda > \lambda_s = \frac{2\pi}{\alpha_s}$ (and nonvanishing phase velocity $0 < c_r < c_s$) are self-excited in the limiting case of infinite Reynolds number. On the other hand, when $\frac{d}{dy}\left(\frac{w'}{T}\right)$ does not vanish for any value of $w > 0$, then except for the "singular" neutral disturbance of zero phase velocity and infinite wave length ($c = 0$ and $\alpha = 0$), all disturbances are damped in the inviscid compressible fluid. This singular neutral disturbance can be regarded as the limiting case of the neutral subsonic disturbance in a real compressible fluid as $R \longrightarrow \infty$.

b. Supersonic Free-Stream Velocity $\left(M_0 > 1\right)$

When the velocity of the free stream is supersonic, the subsonic boundary-layer disturbances must satisfy not only the differential equations and the boundary conditions of the problem but also the physical requirement that $c_r > 1 - \frac{1}{M_0}$. The asymptotic behavior at large Reynolds numbers of the curve of α against R for the neutral subsonic disturbances is determined by the approximate relations (4.1) to (4.4), with the additional restriction that $c > 1 - \frac{1}{M_0}$. As $c \longrightarrow 1 - \frac{1}{M_0}$, $\alpha \longrightarrow 0$ by equation (4.4); therefore $R \longrightarrow \infty$ by equation (4.3). The corresponding value (or values) of z is determined by equation (4.1) as follows:

$$\Phi_1(z) = v(c) = v\left(1 - \frac{1}{M_0}\right) = \frac{-\pi w_1'\left(1 - \frac{1}{M_0}\right)}{T_1}\left[\frac{T^2}{(w')^3}\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c=1-\frac{1}{M_0}} \qquad (4.15)$$

Now from physical considerations, $\frac{d}{dy}\left(\frac{w'}{T}\right) < 0$ for large values of y. Therefore, if $\frac{d}{dy}\left(\frac{w'}{T}\right) = 0$ (changes sign) for some value of $w = c_s > 1 - \frac{1}{M_o}$, then, in general, $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=1-\frac{1}{M_o}} > 0$ and $\Phi_i(z)_{c=1-\frac{1}{M_o}} < 0$ (equation (4.15)). From figure 9, it can be seen that in this case there is only one value of z (z_1, say) corresponding to the value of $\Phi_i(z)$ given by equation (4.15). From equations (4.2) to (4.4), along the lower branch of the curve of α against R for the neutral disturbances,

$$R \approx \frac{{T_1}^{1.76}\left({w_1}'\right)^2 {z_1}^3}{\left(1 - \frac{1}{M_o}\right)^3} \frac{1}{\alpha} \tag{4.16}$$

$$\alpha \approx \frac{{w_1}'\left(1 - \frac{1}{M_o}\right)\sqrt{2M_o}}{T_1 u_1} \sqrt{c - \left(1 - \frac{1}{M_o}\right)} \tag{4.17}$$

and $c \longrightarrow 1 - \frac{1}{M_o}$ at large Reynolds numbers (fig. 4(k)). The upper branch of the curve in this case is given by equations (4.11) and (4.12), or by equations (4.13) and (4.12) if $\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1$ vanishes, with $c \longrightarrow c_s > 1 - \frac{1}{M_o}$ and $\alpha \longrightarrow \alpha_s \neq 0$.

If $\frac{d}{dy}\left(\frac{w'}{T}\right)$ vanishes for $w = 1 - \frac{1}{M_o}$, then $z \longrightarrow \infty$ as $R \longrightarrow \infty$ along the upper branch of the curve of α against R for the neutral disturbances, and $\Phi_1(z) \longrightarrow \frac{w_1'}{\sqrt{2\alpha \frac{R}{\nu_c} c^3}}$. Now $\alpha \longrightarrow 0$ as $c \longrightarrow 1 - \frac{1}{M_o}$ in this case also (equation (4.17) with $u_1 = 1.0$) so that

$$R \approx \frac{2\left(w_1'\right)^{12}}{\pi^2 T_1^{4.24}} \frac{M_o^2}{1 - \frac{1}{M_o}} \frac{1}{\left\{\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1\right\}^2} \frac{1}{\alpha^5} \tag{4.18}$$

Along the lower branch of the curve of α against R at large Reynolds numbers, α, R, and c are connected by equations (4.16) and (4.17), with $z_1 = 2.29$ and $u_1 = 2.29$. In spite of the fact that $\frac{d}{dy}\left(\frac{w'}{T}\right) = 0$ for $w = 1 - \frac{1}{M_o}$, a neutral sonic disturbance $\left(c = 1 - \frac{1}{M_o}\right)$ of finite wave length does not exist in the inviscid fluid unless $K_1(c) = \int_0^\infty \left[\frac{T}{(w - c)^2} - M_o^2\right] dy$ is positive. (See section 10 of reference 8.) Calculation shows that $K_1(c)$ is almost always negative (equation (3.11)); therefore, in general, the sonic disturbance of infinite wave length $(\alpha = 0)$ with constant phase across the boundary layer exists only in the inviscid fluid $(R \longrightarrow \infty)$.

If $\frac{d}{dy}\left(\frac{w'}{T}\right)$ does not vanish for any value of $w \geqq 1 - \frac{1}{M_o}$, it is certain that $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c=1-\frac{1}{M_o}} < 0$ and by equation (4.15)

$\Phi_1(z)_{c=1-\frac{1}{M_0}} > 0$. When $v_{1-\frac{1}{M_0}} < 0.580$ (approx.), there are two values of z $\left(z_2 \text{ and } z_3, \text{ say, with } z_3 > z_2\right)$ corresponding to the value of $\Phi_1(z)$ given by equation (4.15). (See fig. 9.) Along the two asymptotic branches of the curve of α against R for the neutral disturbances, α, R, and c are connected by relations of the form of equations (4.16) and (4.17), with z and u replaced by z_2 and u_2, respectively, along the lower branch and by z_3 and u_3, respectively, along the upper branch. At a given value of the Mach number, the value of $v_{1-\frac{1}{M_0}}$ is controlled by the thermal conditions at the solid surface. (See section 6.) When these conditions are such that $v_{1-\frac{1}{M_0}} \approx 0.580$, then $z_2 = z_3$, and the two asymptotic branches of the curve of α against R for the neutral disturbances coincide. When $v_{1-\frac{1}{M_0}} \geqq 0.580$ (approx.), it is impossible for a neutral or a self-excited subsonic disturbance to exist in the laminar boundary layer of a viscous conductive gas at any value of the Reynolds number. In other words, if $v_{1-\frac{1}{M_0}} \geqq 0.580$ (approx.), the laminar boundary layer is stable at all values of the Reynolds number. (Of course, in any given case, the critical conditions beyond which only damped subsonic disturbances exist can be calculated more accurately from the relations (2.28) and (2.29). See section 5 on minimum critical Reynolds number.)

The preceding conclusion can also be deduced, at least qualitatively, from the results of a study of the energy balance for a neutral subsonic disturbance in the laminar boundary layer. A neutral subsonic disturbance can exist only when the destabilizing effect of viscosity near the solid surface, the damping effect of viscosity in the fluid, and the energy transfer between mean flow and disturbance in the vicinity of the inner "critical layer" all balance out to give a zero (average) net rate of change of the energy of the disturbance. (See Schlichting's discussion for incompressible fluid in reference 4.) In reference 8 it is shown that the sign and magnitude of the phase shift in $u^{*\prime}$ through the inner "critical layer" at $w = c$ is determined by the sign and magnitude of the quantity $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c}$. The corresponding apparent shear stress $\tau_c{}^* = -\overline{\rho^*}\ \overline{u^{*\prime}v^{*\prime}}$, which is zero for $w < c$ in the inviscid compressible fluid, is given by the following expression for $w > c$ (reference 8).

$$\tau_c{}^* = \overline{\rho_o{}^*}\left(\overline{u_o{}^*}\right)^2 \frac{\alpha}{2}\pi \frac{T_c{}^2}{\left(w_c{}'\right)^3}\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c} \tag{4.19}$$

If the quantity $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c}$ is negative, the mean flow absorbs energy from the disturbance; if $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c}$ is positive, energy passes from the mean flow to the disturbance. In the real compressible fluid, the thickness of the inner critical layer in which the viscous forces are important is of the order of $\frac{1}{\left(\alpha\frac{R}{\nu_c}\right)^{1/3}}$, and the phase shift in $u^{*\prime}$ is actually brought about by the effects of viscous diffusion (of the quantity $\rho\frac{dw}{dy}$) through this layer.

As shown by Prandtl (reference 12), the destabilizing effect of viscosity near the solid surface is to shift the phase of the "frictional" component $u_{fr}{}^{*\prime}$ of the disturbance velocity against the phase of the "frictionless" or "inviscid" component $u_{inv}{}^{*\prime}$ in a thin layer of fluid of thickness of the order of $\sqrt{\frac{1}{\alpha c\frac{R}{\nu_1}}}$. By continuity, the associated normal component $v_{fr}{}^{*\prime}$ is of the order of $\left|\frac{\partial u^{*\prime}}{\partial x^*}\right|\sqrt{\frac{1}{\alpha c\frac{R}{\nu_1}}} \approx \left|u_{inv}{}^{*\prime}\right|_1\sqrt{\frac{\alpha}{c\frac{R}{\nu_1}}}$. (It was shown in part 1 of reference 8, that for large values of αR the "frictional" components of the disturbance also satisfy the continuity relation, $\frac{\partial u^{*\prime}}{\partial x^{*}} + \frac{\partial v^{*\prime}}{\partial y^{*}} = 0$ in the compressible fluid.) The corresponding apparent shear stress $\tau_1^* = \overline{-\rho_1^*}\ \overline{u^{*\prime}v^{*\prime}}$ is given by the expression

$$\tau_1^* \approx \overline{\rho_o^*}\left(\overline{u_o^*}\right)^2 \frac{\overline{\rho_1^*}}{\overline{\rho_o^*}}\left[\left(\frac{u_{inv}^{*\prime}}{\overline{u_o^*}}\right)_1\right]^2\sqrt{\frac{\alpha}{c\frac{R}{\upsilon_1}}} \tag{4.20}$$

But from equations (2.11)

$$\left|\left(\frac{u_{inv}^{*\prime}}{\overline{u_o^*}}\right)_1\right| \approx \left|\frac{T_1}{T_1 - M_o^2c^2}\varphi_{21}'\right| = \frac{T_1}{c} \tag{4.21}$$

and

$$\tau_1^* \approx \overline{\rho_o^*}\left(\overline{u_o^*}\right)^2 \frac{T_1}{c^2}\sqrt{\frac{\alpha}{c\frac{R}{\upsilon_1}}} \tag{4.22}$$

Since the shear stress associated with the destabilizing effect of viscosity near the solid surface and the shear stress near the critical layer act roughly throughout the same region of the fluid, the ratio of the rates of energy transferred $\left(\text{approximately } \int_0^h \tau^* \frac{du^*}{dy^*}\,dy^*\right)$ by the two physical processes is

$$\left|\frac{E_c^*}{E_1^*}\right| \sim \left|\frac{T_c^*}{T_1^*}\right| \approx \frac{\pi}{2}\frac{w_1'c}{T_1}\frac{T_c^{\,2}}{(w_c')^3}\left|\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c}\right| z^{3/2} \tag{4.23}$$

$$= \frac{1}{2}\left|v(c)\right| z^{3/2}$$

where

$$z^3 \approx \alpha \frac{R}{\nu_c}\frac{c^3}{(w_1')^2}$$

If the quantity $\frac{d}{dy}\left(\frac{w'}{T}\right)$ is negative and sufficiently large when $w = c_1$, say, then the rate at which energy is absorbed by the mean flow near the inner "critical layer" plus the rate at which the energy of the disturbance is dissipated by viscous action more than counterbalances the rate at which energy passes from the mean flow to the disturbance because of the destabilizing effect of viscosity near the solid surface. Consequently, a neutral subsonic disturbance with the phase velocity $c \gtreqq c_1$ does not exist; in fact, all subsonic disturbances for which $c \geq c_1$ are damped. When $M_o < 1$, there is always a range of values of phase velocity $0 \leqq c \leqq c_o$ for which the ratio $\left|\frac{E_c^*}{E_1^*}\right|$, given by equation (4.22), is small enough for neutral (and self-excited) subsonic disturbances to exist for Reynolds numbers greater than a certain critical value. When $M_o > 1$, however, because of the physical requirement that $c > 1 - \frac{1}{M_o} > 0$, the possibility exists that for certain thermal conditions at the solid surface the quantity $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c}$ is always sufficiently large negatively (and therefore $\left|\frac{E_c^*}{E_1^*}\right|$ is sufficiently large) so that only damped subsonic disturbances exist at all Reynolds numbers. Of course, if $\frac{d}{dy}\left(\frac{w'}{T}\right)$ vanishes for some value of $w \geqq 1 - \frac{1}{M_o}$, it is certain that $v(c) < 0.580$ for some range of values of the phase velocity $1 - \frac{1}{M_o} \leqq c \leqq c_o$. In that case, neutral and self-excited subsonic disturbances always exist for $R > R_{cr_{min}}$ and the flow is always unstable at sufficiently high Reynolds numbers, in accordance with Heisenberg's criterion as extended to the compressible fluid (section 2).

A discussion of the significance of these results is reserved for a later section (section 6) in which the behavior of the quantity $\frac{d}{dy}\left(\frac{w'}{T}\right)$ will be related directly to the thermal conditions at the solid surface and the free-stream Mach number.

5. CRITERION FOR THE MINIMUM CRITICAL REYNOLDS NUMBER

The object of the stability analysis is not only to determine the general conditions under which the laminar boundary layer is unstable at sufficiently high Reynolds numbers but also, if possible, to obtain some simple criterion for the limit of stability of the flow (minimum critical Reynolds number) in terms of the local distribution of mean velocity and mean temperature across the boundary layer. For plane Couette motion (linear velocity profile) and plane Poiseuille motion (parabolic velocity profile) in an incompressible fluid, Synge (reference 13) was able to prove rigorously that a minimum critical Reynolds number actually exists below which the flow is stable. His proof applies also to the laminar boundary layer in an incompressible fluid, with only a slight modification (reference 5, part III). Such a proof is more difficult to give for the laminar boundary layer in a viscous conductive gas; however, the existence, in general, of a minimum critical Reynolds number can be inferred from purely physical considerations. A study of the energy balance for small disturbances in the laminar boundary layer shows that the ratio of the rate of viscous dissipation to the rate of energy transfer near the critical layer is $1/R$ for a disturbance of given wave length while the energy transfer associated with the destabilizing action of viscosity near the solid surface bears the ratio $1/\sqrt{R}$ to the energy transfer near the critical layer. Thus, the effects of viscous dissipation will predominate at sufficiently low Reynolds numbers and all subsonic disturbances will be damped. The two distinct asymptotic branches of the curve of α against R for the neutral disturbances at large Reynolds numbers must join eventually (section 4) and the flow is stable for all Reynolds numbers less than a certain critical value.

An estimate of the value of $R_{cr_{min}}$, which will serve as a stability criterion, is obtained by taking the phase velocity c to have the maximum possible value c_o for a neutral subsonic disturbance, that is, for $c > c_o$ all subsonic disturbances are damped. This condition is very nearly equivalent to the condition that αR be a minimum, which was employed by Lin for the case of the incompressible fluid (p. 285 of reference 5, part III). The condition $c = c_o$ occurs when $\Phi_i(z)$ is a maximum; that is, when $\Phi_i(z) = 0.580$, $z_o = 3.22$ and $\Phi_r(z_o) = 1.48$ (fig. 9). The corresponding value of $c = c_o$ can be calculated from the relations (2.19) to (2.22). Neglecting terms in λ^2 (λ is usually very small) and taking $u = 1.50$ gives

$$\Phi_i(z) \approx \left[1 - 2\lambda(c)\right] v(c) \tag{5.1}$$

where

$$v(c) = -\pi \frac{w_1'c}{T_1} \left[\frac{T^2}{(w')^3} \frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c} \tag{5.2}$$

and

$$\lambda(c) = \frac{w_1'(y_c - y_1)}{c} - 1 \tag{5.3}$$

It is only necessary to plot the quantity $(1 - 2\lambda)v$ against c for a given laminar boundary-layer flow and find the value of $c = c_o$ for which $(1 - 2\lambda)v = 0.580$. The corresponding value of αR is determined from the relation

$$\alpha R = \left[T(c_o)\right]^{1.76} (w_1')^2 \left(\frac{z_o}{c_o}\right)^3 \tag{5.4}$$

and this value of αR is very close to the minimum value of αR. A rough estimate of the value of α for $c = c_o$ is given by the following relation (equation (2.27)):

$$\alpha \approx w_1' c_o \sqrt{1 - M_o^2(1 - c_o)^2} \tag{5.5}$$

This estimated value of α is, in general, too small. The following estimate of $R_{cr_{min}}$ is obtained by making an approximate allowance for this discrepancy and by taking round numbers:

$$R_{cr_{min}} \approx \frac{25 \left[T(c_o)\right]^{1.76} w_1'}{c_o^4 \sqrt{1 - M_o^2(1 - c_o)^2}} \tag{5.6}$$

or

$$R_{\theta_{cr_{min}}} \approx \frac{17 \left[T(c_o)\right]^{1.76} \left(\frac{\partial w}{\partial \eta}\right)_1}{c_o^4 \sqrt{1 - M_o^2(1 - c_o)^2}} \tag{5.7}$$

For zero pressure gradient, the slope of the velocity profile at the surface $\left(\frac{\partial w}{\partial \eta}\right)_1$ is given very closely by (appendix B)

$$\left(\frac{\partial w}{\partial \eta}\right)_1 = \frac{\left(\frac{\partial w}{\partial \eta}\right)_{1_B}}{T_1}$$

$$= \frac{0.332}{T_1}$$

Therefore

$$R_{\theta_{cr_{min}}} \approx \frac{6}{T_1} \frac{\left[T(c_o)\right]^{1.76}}{c_o \sqrt[4]{1 - M_o^2(1 - c_o)^2}} \tag{5.8}$$

The expression (5.8) is useful as a rough criterion for the dependence of $R_{\theta_{cr_{min}}}$ on the local distribution of mean velocity and mean temperature across the boundary layer. It is immediately evident that $R_{\theta_{cr_{min}}} \longrightarrow \infty$ when $c_o \longrightarrow 1 - \frac{1}{M_o}$. When $\left[(1 - 2\lambda)v\right]_{c=1-\frac{1}{M_o}} \geqq 0.580$, <u>the laminar boundary layer is stable at all values of the Reynolds number</u>. (This condition is an improvement on the stability condition $v_{1-\frac{1}{M_o}} \geqq 0.580$ (approx.) stated in section 4.)

In the following tables and in figures 5 and 6(a) the estimated values of $R_{\theta_{cr_{min}}}$ given by equation (5.8) can be compared with the values of $R_{\theta_{cr_{min}}}$ taken from the calculated curves of α_θ against R_θ for the neutral disturbances. For the insulated surface, the values are

M_o	c_o	$T(c_o)$	$R_{\theta_{cr_{min}}}$ (est.)	$R_{\theta_{cr_{min}}}$ (fig. 4)
0	0.4186	1.0000	195	150
.50	.4400	1.0408	170	136
.70	.4600	1.0782	150	126
.90	.4850	1.1254	129	115
1.10	.5139	1.1803	109	104
1.30	.5450	1.2406	92	92

For the noninsulated surface when $M_o = 0.70$, the values are

T_1	c_o	$T(c_o)$	$R_{\theta_{cr_{min}}}$ (est.)	$R_{\theta_{cr_{min}}}$ (fig. 4)
0.70	0.1872	0.7712	5377	5150
.80	.2619	.8716	1463	1440
.90	.3394	.9562	524	523
1.25	.5194	1.1449	89	63

The expression (5.8) for $R_{\theta_{cr_{min}}}$ gives the correct order of magnitude and the proper variation of the stability limit with Mach number and with surface temperature at a given Mach number.

The form of the criterion for the minimum critical Reynolds number (equation (5.8)) and the results of the detailed stability calculations for several representative cases (figs. 3 and 4) show that the distribution of the product of the density and the vorticity $\rho\frac{dw}{dy}$ across the boundary layer largely determines the limits of stability of laminar boundary-layer flow. The fact that the "proper" Reynolds number that appears in the boundary-layer stability calculations is based on the kinematic viscosity at the inner critical layer (where the viscous forces are important) rather than in the free stream also enters the problem, but it amounts only to a numerical and not a qualitative change when the usual Reynolds number based on free-stream kinematic viscosity is finally computed. Whether the value of $R_{\theta_{cr_{min}}}$ for a given laminar boundary-layer flow is larger or smaller than the value of $R_{\theta_{cr_{min}}}$ for the Blasius flow, for example, is determined entirely by the distribution of $\rho\frac{dw}{dy}$ across the boundary layer. If the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ is negative and large near the solid surface so that the quantity $(1 - 2\lambda)v(c)$ reaches the value 0.580 when the value of $c = c_o$ is less than 0.4186, the flow is relatively more stable than the Blasius flow. If the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ is positive near the solid surface, so that $(1 - 2\lambda)v(c) = 0.580$ when $w(\text{or } c) > 0.4186$, the flow is relatively less stable than the Blasius flow. Thus, the question of the relative influence on $R_{\theta_{cr_{min}}}$ of the kinematic viscosity at the inner critical layer and the distribution of $\rho\frac{dw}{dy}$ across the boundary layer, which remained open in the concluding discussions of reference 8, is now settled.

The physical basis for the predominant influence on $R_{\theta_{cr_{min}}}$ of the distribution of $\rho\frac{dw}{dy}$ across the boundary layer is to be found in a study of the energy balance for a subsonic boundary-layer disturbance (section 4). The distribution of $\rho\frac{dw}{dy}$ determines the maximum possible value of the phase velocity c_o or the maximum possible distance of the inner critical layer from the solid surface for a neutral subsonic disturbance. The greater the distance of the inner critical layer from the solid surface, the greater (relatively) the rate of energy absorbed by the mean flow from the disturbance in the vicinity of the critical layer (equations (4.21) and (4.22)). When c_o is large, therefore, the energy balance for a neutral subsonic disturbance is achieved only when the destabilizing action of viscosity near the solid surface is relatively large or, in other words, when $\frac{1}{\sqrt{\alpha_o\frac{R_o}{\nu_{c_o}}}} \approx c_o^{3/2}$ is large and the Reynolds number R_o, which is very nearly equal to $R_{cr_{min}}$, is correspondingly small. On the other hand, when c_o is small and the inner critical layer is close to the solid surface, the rate at which energy is absorbed from the disturbance near the critical layer is relatively small and the rate at which energy passes to the disturbance near the solid surface, which is of the order of $\frac{1}{\sqrt{\alpha \frac{R}{\upsilon_c}}}$, is also relatively small for energy balance; consequently $R_{cr_{min}}$ is large.

6. PHYSICAL SIGNIFICANCE OF RESULTS OF STABILITY ANALYSIS

a. General

From the results obtained in the present paper and in reference 8, it is clear that the stability of the laminar boundary layer in a compressible fluid is governed by the action of both viscous and inertia forces. Just as in the case of an incompressible fluid, the stability problem cannot be understood unless the viscosity of the fluid is taken into account. Thus, whether or not a laminar boundary-layer flow is unstable in the inviscid compressible fluid $(R \rightarrow \infty)$, that is, whether or not the product of the density and the vorticity $\rho \frac{dw}{dy}$ has an extremum for some value of $w > 1 - \frac{1}{M_o}$, there is always some value of the Reynolds number $R_{cr_{min}}$ below which the effect of viscous dissipation predominates and the flow is stable. On the other hand, at very large Reynolds numbers the influence of viscosity is destabilizing. If the free-stream velocity is subsonic, any laminar boundary-layer flow is unstable at sufficiently high (but finite) Reynolds numbers, whether or not the flow is stable in the inviscid fluid when only the inertia forces are considered.

The action of the inertia forces is more decisive for the stability of the laminar boundary layer if the free-stream velocity is supersonic. Because of the physical requirement that the relative phase velocity $(c - 1)$ of the boundary-layer disturbances must be subsonic, it follows that $c > 1 - \frac{1}{M_o} > 0$ and the quantity $\left[\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)\right]_{w=c}$ can be large enough negatively under certain conditions so that the stabilizing action of the inertia forces near the inner critical layer (where $w = c > 0$) is not overcome by the destabilizing action of viscosity near the solid surface. In that case, undamped disturbances cannot exist in the fluid, and the flow is stable at all values of the Reynolds number.

Regardless of the free-stream velocity, the distribution of the product of the density and the vorticity $\rho\frac{dw}{dy}$ across the boundary layer determines the actual limit of stability, or the minimum critical Reynolds number, for laminar boundary-layer flow in a viscous conductive gas (equation (9.8)). Since the distribution of $\rho\frac{dw}{dy}$ across the boundary layer in turn is determined by the free-stream Mach number and the thermal conditions at the solid surface, the effect of these physical parameters on the stability of laminar boundary-layer flow is readily evaluated.

b. Effect of Free-Stream Mach Number and Thermal Conditions at Solid Surface on Stability of Laminar Boundary Layer

The distribution of mean velocity and mean temperature (and therefore of $\rho\frac{dw}{dy}$) across the laminar boundary layer in a viscous conductive gas is strongly influenced by the fact that the viscosity of a gas increases with the temperature. (For most gases, $\mu \propto T^m$ ($m = 0.76$ for air) over a fairly wide temperature range.) When heat is transferred to the fluid through the solid surface, the temperature and viscosity near the surface both decrease along the outward normal, and the fluid near the surface is more retarded by the viscous shear than the fluid farther out from the surface - as compared with the isothermal Blasius flow. The velocity profile therefore always possesses a point of inflection (where $w'' = 0$) when heat is added to the fluid through the solid surface, provided there is no pressure gradient in the direction of the main flow. Since $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right) = \frac{w''}{T} - \frac{w'T'}{T^2}$, the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ vanishes and $\rho\frac{dw}{dy}$ has an extremum at some point in the fluid. On the other hand, if heat is withdrawn from the fluid through the solid surface, $\frac{\partial T}{\partial y}$ and $\frac{\partial \mu}{\partial y}$ are both positive near the surface and the fluid near the surface is less retarded than the fluid farther out - as compared with the Blasius flow. The velocity profile is therefore more convex near the surface than the Blasius profile.

As pointed out in section 11 of reference 8, the influence of the variable viscosity on the behavior of the product of the density and the vorticity $\rho\frac{dw}{dy}$ can be seen directly from the equations of motion for the mean flow. When there is no pressure gradient in the direction of the main flow, the fluid acceleration vanishes at the solid surface, or

$$\left(\frac{\partial \overline{\tau^*}}{\partial y^*}\right)_1 = \left[\frac{\partial}{\partial y^*}\left(\overline{\mu_1^*}\frac{\partial \overline{u^*}}{\partial y^*}\right)\right]_1 = 0 \tag{6.1}$$

and

$$\left(\frac{\partial^2 \overline{u^*}}{\partial y^{*2}}\right)_1 = -\frac{1}{\overline{\mu_1^*}}\left(\frac{\partial \overline{\mu^*}}{\partial y^*}\right)_1\left(\frac{\partial \overline{u^*}}{\partial y^*}\right)_1 = -\frac{m}{T_1}\left(\frac{\partial \overline{T^*}}{\partial y^*}\right)_1\left(\frac{\partial \overline{u^*}}{\partial y^*}\right)_1 \tag{6.2}$$

Thus, when heat is added to the fluid through the solid surface $\left(T_1' < 0\right)$, $\left(\frac{\partial^2 \overline{u^*}}{\partial y^{*2}}\right)_1$ is positive, and the velocity profile is concave near the surface and possesses a point of inflection for some value of $w > 0$; when heat is withdrawn from the fluid $\left(T_1' > 0\right)$, $\left(\frac{\partial^2 \overline{u^*}}{\partial y^{*2}}\right)_1$ is negative, and the velocity profile is more convex near the surface than the Blasius profile.

The behavior of the quantity $\frac{\partial}{\partial y^*}\left(\frac{1}{\overline{T^*}}\frac{\partial \overline{u^*}}{\partial y^*}\right) = \frac{d}{dy^*}\left(\rho^*\frac{d\overline{u^*}}{dy^*}\right)$ is parallel to that of $\frac{\partial^2 \overline{u^*}}{\partial y^{*2}}$. From equation (6.2), in nondimensional form,

$$\left[\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)\right]_1 = \left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_1 = -\frac{m+1}{T_1{}^2}T_1{}'w_1{}' \qquad (6.3)$$

Differentiating the dynamic equations once and making use of the energy equation gives the following expression for $\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1$ (appendix D):

$$\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1 = \sigma(m+1)(\gamma-1)M_0{}^2\,\frac{(w_1{}')^3}{T_1{}^2} + 2(m+1)^2\,w_1{}'\,\frac{(T_1{}')^2}{T_1{}^3} \qquad (6.4)$$

Thus, for zero pressure gradient, $\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1$ is always positive. Now, if the surface is insulated, the quantity $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_1$ vanishes, but $\left[\frac{d^2}{dy}\left(\frac{w'}{T}\right)\right]_1 > 0$ and $\frac{d}{dy}\left(\frac{w'}{T}\right)$ and $\frac{w'}{T}$ both increase with distance from the solid surface. Since $\frac{w'}{T} \longrightarrow 0$ far from the solid surface, $\frac{w'}{T}$ has a maximum and $\frac{d}{dy}\left(\frac{w'}{T}\right)$ vanishes for some value of $w > 0$. If heat is added to the fluid through the solid surface $(T_1{}' < 0)$, $\frac{d}{dy}\left(\frac{w'}{T}\right)$ is already positive at the surface, and since $\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1 > 0$, the quantity $\frac{d}{dy}\left(\frac{w'}{T}\right)$ vanishes at a point in the fluid which is farther from the surface than for an insulated boundary at the same Mach number (figs. 3(a) and (b)). Consequently, the value of $c = c_0$ for which the function

$(1 - 2\lambda)v(c) = -\pi(1 - 2\lambda)\dfrac{w_1'c}{T_1}\left[\dfrac{T^2}{(w')^3}\dfrac{d}{dy}\left(\dfrac{w'}{T}\right)\right]_{w=c}$ reaches the value 0.580 is larger than the value for the insulated surface. By equation (5.8), the effect of adding heat to the fluid through the solid surface is to reduce $R_{\theta_{cr_{min}}}$ and to destabilize the flow, as compared with the flow over an insulated surface at the same Mach number (fig. 6).

If heat is withdrawn from the fluid through the solid surface, $T_1' > 0$ and $\left[\dfrac{d}{dy}\left(\dfrac{w'}{T}\right)\right]_1$ is negative. In fact, if the rate of heat transfer is sufficiently large, the quantity $\dfrac{d}{dy}\left(\dfrac{w'}{T}\right)$ does not vanish within the boundary layer (fig. 3(b)). The value of $c = c_o$ for which the function $(1 - 2\lambda)v(c)$ reaches the value 0.580 is smaller than for an insulated surface at the same Mach number, and by equation (5.8), the effect of withdrawing heat from the fluid through the solid surface is to increase $R_{\theta_{cr_{min}}}$ and to stabilize the flow, as compared with the flow over an insulated surface at the same Mach number (fig. 6). When the velocity of the free stream at the "edge" of the boundary layer is supersonic, the laminar boundary layer is completely stabilized if the rate at which heat is withdrawn through the solid surface reaches or exceeds a critical value that depends only on the Mach number, the Reynolds number, and the properties of the gas. The critical rate of heat transfer is that for which the quantity $\dfrac{d}{dy}\left(\dfrac{w'}{T}\right)$ is sufficiently large negatively near the surface (see equation (6.3)) so that $(1 - 2\lambda)v(c) = 0.580$ when $c = c_o = 1 - \dfrac{1}{M_o}$ (sections 4 and 5). Although detailed stability calculations for supersonic flow over a noninsulated surface have not been carried out, the function $(1 - 2\lambda)v(c)$ has been computed for noninsulated surfaces at $M_o = 1.30, 1.50, 2.00, 3.00$, and 5.00 by a rapid approximate method (appendix C). The corresponding estimated values of $R_{\theta_{cr_{min}}}$ were calculated from equation (5.8), and in figure 7 these values are plotted against T_1, the ratio of surface temperature (deg abs.) to free-stream temperature (deg abs.). At any given Mach number greater than unity the value of $R_{\theta_{cr_{min}}}$ increases rapidly as $c_o \longrightarrow 1 - \frac{1}{M_o}$; when c_o differs only slightly from $1 - \frac{1}{M_o}$, the stability of the laminar boundary layer is extremely sensitive to thermal conditions at the solid surface. At each value of $M_o > 1$, there is a critical value of the temperature ratio $T_{1_{cr}}$ for which $R_{\theta_{cr_{min}}} \longrightarrow \infty$. If $T_1 \leqq T_{1_{cr}}$, the laminar boundary layer is stable at all Reynolds numbers. The difference between the stagnation-temperature ratio and the critical-surface-temperature ratio, which is related to the heat-transfer coefficient, is plotted against Mach number in figure 8. Under free-flight conditions, for Mach numbers greater than some critical Mach number that depends largely on the altitude, the value of $T_s - T_{1_{cr}}$ is within the order of magnitude of the difference between stagnation temperature and surface temperature that actually exists because of heat radiation from the surface (references 14 and 15). In other words, the critical rate of heat withdrawal from the fluid for laminar stability is within the order of magnitude of the calculated rate of heat conduction through the solid surface which balances the heat radiated from the surface under equilibrium conditions. The calculations in appendix E show that this critical Mach number is approximately 3 at 50,000 feet altitude and approximately 2 at 100,000 feet altitude. Thus, for $M_o > 3$ (approx.) at 50,000 feet altitude and $M_o > 2$ (approx.) at 100,000 feet altitude, the laminar boundary-layer flow for thermal equilibrium is completely stable in the absence of an adverse pressure gradient in the free stream.

When there is actually no heat conduction through the solid surface, the limit of stability of the laminar boundary layer depends only on the free-stream Mach number, that is, on the extent of the "aerodynamic heating" $\left(\text{of the order of } \overline{u_1^*}\left(\frac{\partial \overline{u^*}}{\partial y^*}\right)^2\right)$ near the solid surface. A good indication of the influence of the free-stream Mach number on the distribution of $\rho\frac{dw}{dy}$ across the boundary layer for an insulated surface is obtained from a rough estimate of the location of the point at which $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ reaches a positive maximum $\left(\text{or } \frac{d^2}{dy^2}\left(\rho\frac{dw}{dy}\right) \text{ vanishes}\right)$. Differentiating the dynamic equations of mean motion twice and making use of the energy and continuity equations yields the following result for an insulated surface:

$$\left[\frac{d^3}{dy^3}\left(\frac{w'}{T}\right)\right]_1 = -\frac{b^2}{2}\frac{\left(w_1'\right)^2}{T_1^{m+2}} \tag{6.5}$$

where $b = \delta\sqrt{\frac{\overline{u_o^*}}{\overline{\nu_o^*}\ x^*}}$. From equations (6.4) and (6.5) the value of c at which $\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)$ vanishes, or $\frac{d}{dy}\left(\frac{w'}{T}\right)$ reaches a maximum, is given roughly for air by

$$c \approx \frac{w_1'\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1}{-\left[\frac{d^3}{dy^3}\left(\frac{w'}{T}\right)\right]_1} \approx \frac{M_o^{\ 2}}{T_1^{\ 2-m}} = \frac{M_o^{\ 2}}{\left(1 + 0.2025M_o^{\ 2}\right)^{1.24}} \tag{6.6}$$

in which $w_1' \approx \frac{b(0.3320)}{T_1}$ (appendix B). In other words, the point in the fluid at which $\frac{d}{dy}\left(\frac{w'}{T}\right)$ attains a maximum moves farther out from the surface as the Mach number is increased - at least in the range $0 \leqq M_o \leqq 4.5$ (approx.); therefore the value of c for which $\frac{d}{dy}\left(\frac{w'}{T}\right)$ vanishes and the value of $c = c_o$ for which $(1 - 2\lambda)v(c)$ reaches the value 0.580 both increase with the Mach number (fig. 3(a)). By equation (5.8), the value of $R_{\theta_{cr_{min}}}$ for the laminar boundary-layer flow over an insulated surface decreases as the Mach number increases and the flow is destabilized, as compared with the Blasius flow (fig. 5).

c. Results of Detailed Stability Calculations for Insulated and Noninsulated Surfaces

From the results of the detailed stability calculations for several representative cases (figs. 4 to 6), a quantitative estimate can be made of the effect of free-stream Mach number and thermal conditions at the solid surface on the stability of laminar boundary-layer flow. For the insulated surface, the value of $R_{\theta_{cr_{min}}}$ is 92 when $M_o = 1.30$ as compared with a value of 150 for the Blasius flow. For the noninsulated surface at $M_o = 0.70$, the value of $R_{\theta_{cr_{min}}}$ is 63 when $T_1 = 1.25$ (heat added to fluid), $R_{\theta_{cr_{min}}} = 126$ when $T_1 = 1.10$ (insulated surface), and $R_{\theta_{cr_{min}}} = 5150$ when $T_1 = 0.70$ (heat withdrawn from fluid). Since $R_{x^*} \approx 2.25 R_\theta^2$, (the value of $\theta\sqrt{\frac{u_o^*}{\nu_o^* x^*}}$, which is proportional to the skin-friction coefficient, differs only slightly from the Blasius value of 0.6667) the effect of the thermal conditions at the solid surface on R_{x^*} is even more pronounced. The value of R_{x^*} is 60×10^6 when $T_1 = 0.70$ and $M_o = 0.70$, as compared with a value of 51×10^3 for the Blasius flow $\left(T_1 = 1 \text{ and } M_o = 0\right)$. For the insulated surface the value of $R_{x^*_{cr_{min}}}$ declines from the Blasius value for $M_o = 0$ to a value of 19×10^3 at $M_o = 1.30$. The extreme sensitivity of the limit of stability of the laminar boundary layer to thermal conditions at the solid surface when $T_1 < 1$ is accounted for by the fact that c_o is small when $T_1 < 1$ and $M_o < 1$ (or M_o is not much greater than unity) and $R_{\theta_{cr_{min}}} \approx \frac{1}{c_o^4}$ (equation (5.8)). Small changes in c_o, therefore, produce large changes in $R_{\theta_{cr_{min}}}$. In addition, when $T_1 < 1$, small changes in the thermal conditions at the solid surface produce appreciable changes in $\frac{d}{dy}\left(\frac{w'}{T}\right)$ (equation (6.3)) and, therefore, in the value of c_o.

Not only is the value of $R_{\theta_{cr_{min}}}$ affected by the thermal conditions at the solid surface and by the free-stream Mach number but the entire curve of α_θ against R_θ for the neutral disturbances is also affected. (See figs. 4(k) and 4(l).) When the surface is insulated (and $M_o \neq 0$), or heat is added to the fluid $\left(T_1 = 1.25\right)$, $\alpha_\theta \longrightarrow \alpha_s \neq 0$ as $R_\theta \longrightarrow \infty$ along the upper branch of the curve of neutral stability. In other words, there is a finite range of unstable wave lengths even in the limiting case of an infinite Reynolds number (inviscid fluid). However, $\alpha \longrightarrow 0$ as $R_\theta \longrightarrow \infty$ for the Blasius flow, or when heat is withdrawn from the fluid. This behavior is in complete agreement with the results obtained in section 4 and in reference 8.

A comparison between the curves of α_θ against R_θ for $T_1 = 1.25$ and $T_1 = 0.70$ at $M_o = 0.70$ shows that withdrawing heat from the fluid not only stabilizes the flow by increasing $R_{\theta_{cr_{min}}}$ but also greatly reduces the range of unstable wave numbers $\left(\alpha_\theta\right)$. On the other hand, the addition of heat to the fluid through the solid surface greatly increases the range of unstable wave numbers.

It should also be noted that for given values of α_θ, c, and R_θ the time frequencies of the boundary-layer disturbances in the high-speed flow of a gas are considerably greater than the frequencies of the familiar Tollmien waves observed in low-speed flow. The actual time frequency n^* expressed nondimensionally is as follows:

$$\frac{n^* \, \overline{\nu_o^*}}{\left(\overline{u_o^*}\right)^2} = \frac{c\alpha_\theta}{2\pi R_\theta}$$

For given values of c, α_θ, and R_θ the frequency increases as the square of the free-stream velocity.

d. Instability of Laminar Boundary Layer and

Transition to Turbulent Flow

The value of $R_{\theta_{cr_{min}}}$ obtained from the stability analysis for a given laminar boundary-layer flow is the value of the Reynolds number at which self-excited disturbances first appear in the boundary layer. As Prandtl (reference 12) carefully pointed out, these initial disturbances are not turbulence, in any sense, but slowly growing oscillations. The value of the Reynolds number at which boundary-layer disturbances propagated along the surface will be amplified to a sufficient extent to cause turbulence must be larger than $R_{\theta_{cr_{min}}}$ in any case; for the insulated flat-plate flow at low speeds and with no pressure gradient, the transition Reynolds number $R_{\theta_{tr}}$ is found to be three to seven times as large as the value of $R_{\theta_{cr_{min}}}$ (references 6 and 7). The value of $R_{\theta_{tr}}$ depends not only on $R_{\theta_{cr_{min}}}$ but also on the initial magnitude of the disturbances with the most "dangerous" frequencies (those with greatest amplification), on the rate of amplification of these disturbances, and on the physical process (as yet unknown) by which the quasi-stationary laminar flow is finally destroyed by the amplified oscillations. (See, for example, references 16 and 17.) The results of the stability analysis nevertheless permit certain general statements to be made concerning the effect of free-stream Mach number and thermal conditions at the solid surface on transition. The basis for these statements is summarized as follows:

(1) In many problems of technical interest in aeronautics the level of free-stream turbulence (magnitude of initial disturbances) is sufficiently low so that the origin of transition is always to be found in the instability of the laminar boundary layer. In other words, the value of $R_{\theta_{cr_{min}}}$ is an absolute lower limit for transition.

(2) The effect of the free-stream Mach number and the thermal conditions at the solid surface on the stability limit $\left(R_{\theta_{cr_{min}}}\right)$ is overwhelming. For example, for $M_o = 0.70$, the value of $R_{\theta_{cr_{min}}}$ when $T_1 = 0.70$ (heat withdrawn from fluid) is more than 80 times as great as the value of $R_{\theta_{cr_{min}}}$ when $T_1 = 1.25$ (heat added to fluid).

(3) The maximum rate of amplification of the self-excited boundary-layer disturbances propagated along the surface varies roughly as $1/\sqrt{R_{\theta_{cr_{min}}}}$. (This approximation agrees closely with the numerical results obtained by Pretsch (reference 18) for the case of an incompressible fluid.) The effect of withdrawing heat from the fluid, for example, is not only to increase $R_{\theta_{cr_{min}}}$ and

stabilize the flow in that manner but also to decrease the initial rate of amplification of the unstable disturbances. In other words, for a given level of free-stream turbulence, the interval between the first appearance of self-excited disturbances and the onset of transition is expected to be much longer for a relatively stable flow, for which $R_{\theta_{cr_{min}}}$ is large, than for a relatively unstable flow, for which $R_{\theta_{cr_{min}}}$ is small and the initial rate of amplification is large.

On the basis of these observations, transition is delayed $\left(R_{\theta_{tr}}\right.$ increased$\left.\right)$ by withdrawing heat from the fluid through the solid surface and is advanced by adding heat to the fluid through the solid surface, as compared with the insulated surface at the same Mach number. For the insulated surface, transition occurs earlier as the Mach number is increased, as compared with the flat-plate flow at very low Mach numbers. When the free-stream velocity at the edge of the boundary layer is supersonic, transition never occurs if the rate of heat withdrawal from the fluid through the solid surface reaches or exceeds a critical value that depends only on the Mach number (section 6b and figs. 7 and 8).

A comparison between the results of the present analysis and measurements of transition is possible only when the free-stream pressure gradient is zero or is held fixed while the free-stream Mach number or the thermal conditions at the solid surface are varied. Liepmann and Fila (reference 19) have measured the movement of the transition point on a flat plate at a very low free-stream velocity when heat is applied to the surface. They found by means of the hot-wire anemometer that $R_{x^*_{tr}}$ declined from 5×10^5 for the insulated surface to a value of approximately 2×10^5 for $T_1 = 1.36$ when the level of free-stream turbulence $\sqrt{\frac{\overline{(u^{*\prime})^2}}{(\overline{u_o^*})^2}}$ was 0.17 percent, or to a value of 3×10^5 when $\sqrt{\frac{\overline{(u^{*\prime})^2}}{(\overline{u_o^*})^2}} = 0.05$ percent and $T_1 = 1.40$. The value of $R_{\theta_{tr}}$ declines from 470 (approx.) to 300 (approx.) in the first case and to 365 in the second.

Frick and McCullough (reference 20) observed the variation in the transition Reynolds number when heat is applied to the upper surface of an NACA 65,2-016 airfoil at the nose section alone, at the section just ahead of the minimum pressure station, and for the entire laminar run. When heat is applied only to the nose section, the transition Reynolds number (determined by total-pressure-tube measurements) was practically unchanged. Near the nose, $R_\theta \ll R_{\theta_{cr_{min}}}$ and the strong favorable pressure gradient in the region of the stagnation point stabilizes the laminar boundary layer to such an extent that the addition of heat to the fluid has only a negligible effect. When heat is applied, however, to the section just ahead of the minimum pressure point, where the pressure gradients are moderate, the transition Reynolds number $R_{\theta_{tr}}$ declined to a value of 1190 for $T_1 \approx 1.14$, compared with a value of 1600 for the insulated surface. When heat is applied to the entire laminar run, $R_{\theta_{tr}}$ declined to a value of 1070 for $T_1 \approx 1.14$.

It would be interesting to investigate experimentally the stabilizing effect of a withdrawal of heat from the fluid at supersonic velocities. At any rate, on the basis of the results obtained in the experimental investigations of the effect of heating on transition at low speeds, the results of the stability analysis give the proper direction of this effect.

7. Stability of the Laminar Boundary-Layer Flow of a Gas with a Pressure Gradient in the Direction of the Free Stream

For the case of an incompressible fluid, Pretsch (reference 9) has shown that even with a pressure gradient in the direction of the free stream, the <u>local</u> mean-velocity distribution across the boundary layer completely determines the stability characteristics of the <u>local</u> laminar boundary-layer flow at large Reynolds numbers. From physical considerations this statement should apply also to the compressible fluid, provided only the stability of the flow in the boundary layer is considered and not the possible interaction of the boundary layer and the main "external" flow. Further study is required to settle this question.

If only the local mean velocity-temperature distribution across the boundary layer is found to be significant for laminar stability in a compressible fluid, the criterions obtained in the present paper and in reference 8 are then immediately applicable to laminar boundary-layer gas flows in which there is a free-stream pressure gradient. The quantitative effect of a pressure gradient on laminar stability could be readily determined by means of the approximate estimate of $R_{\theta_{cr_{min}}}$ (equation (5.7)), in terms of the distribution of the quantity $\rho\frac{dw}{dy}$ across the boundary layer. Such calculations (unpublished) have already been carried out by Dr. C. C. Lin of Brown University for the incompressible fluid by means of the approximate estimate of $R_{\delta^*_{cr_{min}}}$ given in reference 5, part III.

In any event, the qualitative effect of a free-stream pressure gradient on the local distribution of $\rho\frac{dw}{dy}$ across the boundary layer is evidently the same in a compressible fluid as in an incompressible fluid. If the effect of the local pressure gradient alone is considered, the velocity distribution across the boundary layer is "fuller" or more convex for accelerated than for uniform flow, and conversely, less convex for decelerated flow. Thus, from the results of the present paper the effect of a negative pressure gradient on the laminar boundary-layer flow of gas is stabilizing, so far as the local mean velocity-temperature distribution is concerned, while a positive pressure gradient is destabilizing. For the incompressible fluid, this fact is well established by the Rayleigh-Tollmien criterion (reference 3), the work of Heisenberg (reference 1) and Lin (reference 5), and a mass of detailed calculations of stability limits from the curves of α against R for the neutral disturbances. These calculations were recently carried out by several German investigators for a comprehensive series of pressure gradient profiles. (See, for example, references 9 and 21.)

Some idea of the relative influence on laminar stability of the thermal conditions at the solid surface and the free-stream pressure gradient is obtained from the equations of mean motion. At the surface,

$$\left(\frac{\partial \overline{\tau^*}}{\partial y^*}\right)_1 = \left[\frac{\partial}{\partial y^*}\left(\overline{\mu_1}^* \frac{\partial \overline{u^*}}{\partial y^*}\right)\right]_1 = \frac{d\overline{p^*}}{dx^*} = -\overline{\rho_o}^* \; \overline{u_o}^* \frac{d\overline{u_o^*}}{dx^*} \tag{7.1}$$

or

$$\left[\frac{d}{dy}\left(\rho \frac{dw}{dy}\right)\right]_1 = -\frac{m+1}{{T_1}^2} T_1' w_1' - \frac{1}{{T_1}^{m+1}} \frac{\delta^2}{\upsilon_o^*} \frac{d\overline{u_o^*}}{dx^*} \tag{7.2}$$

In a region of small or moderate pressure gradients $\left(\left|\frac{\delta^2}{\nu_o^*}\frac{d\overline{u_o}^*}{dx}\right| \leq 2,\right.$ say$\Big)$ the distribution of $\rho\frac{dw}{dy}$ is sensitive to the thermal conditions at the solid surface. For example, the chordwise position of the point of instability of the laminar boundary layer on an airfoil with a flat pressure distribution is expected to be strongly influenced by heat conduction through the surface. (See reference 20.) For the insulated surface, the equations of mean motion yield the following relation (appendix D), which does not involve the pressure gradient explicitly:

$$\left[\frac{d^2}{dy^2}\left(\rho\frac{dw}{dy}\right)\right]_1 = \sigma(m + 1)(\gamma - 1)M_o^2\frac{(w_1')^3}{T_1^2} > 0 \qquad (7.3)$$

The effect of "aerodynamic heating" at the surface opposes the effect of a favorable pressure gradient so far as the distribution of $\rho\frac{dw}{dy}$ across the boundary layer is concerned (equations (7.2) and (7.3)). The relative quantitative influence of these two effects on laminar stability can only be settled by actual calculations of the laminar boundary-layer flow in a compressible fluid with a free-stream pressure gradient. A method for the calculation of such flows over an insulated surface is given in reference 22.

When the local free-stream velocity at the edge of the boundary layer is supersonic, a negative pressure gradient can have a decisive effect on laminar stability. The local laminar boundary-layer flow over an insulated surface, for example, is expected to be completely stable when the magnitude of the local negative pressure gradient reaches or exceeds a critical value that depends only on the local Mach number and the properties of the gas. The critical magnitude of the pressure gradient is that which makes the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ sufficiently large negatively near the surface so that

$$-\left[1 - 2\lambda(c)\right]\pi\frac{w_1'c}{T_1}\left[\frac{T^2}{(w')^3}\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c} = 0.580$$

when $c = 1 - \frac{1}{M_o}$.

It has already been shown in the present paper that when $M_o > 3$ (approx.) the laminar boundary-layer flow with a uniform free-stream velocity is completely stable under free-flight conditions when the solid surface is in thermal equilibrium, that is, when the heat conducted from the fluid to the surface balances the heat radiated from the surface (section 6b). The laminar boundary-layer flow for thermal equilibrium should be completely stable for $M_o > M_s$, say, where $M_s < 3$ if there is a negative pressure gradient in the direction of the free stream. Favorable pressure gradients exist over the forward part of sharp-nosed airfoils and bodies of revolution moving at supersonic velocities, and the limits of stability $\left(R_{\theta_{cr_{min}}}\right)$ of the laminar boundary layer should be calculated in such cases.

CONCLUSIONS

From a study of the stability of the laminar boundary layer in a compressible fluid, the following conclusions were reached:

1. In the compressible fluid as in the incompressible fluid, the influence of viscosity on the laminar boundary-layer flow of a gas is destabilizing at very large Reynolds numbers. If the free-stream velocity is subsonic, any laminar boundary-layer flow of gas is unstable at sufficiently high Reynolds numbers.

2. Regardless of the free-stream Mach number, if the product of the mean density and the mean vorticity has an extremum $\left(\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)\right.$ vanishes$\left.\right)$ for some value of $w > 1 - \frac{1}{M_o}$ (where w is the ratio of mean velocity component parallel to the surface to the free-stream velocity, and where M_o is the free-stream Mach number) the flow is unstable at sufficiently high Reynolds numbers.

3. The actual limit of stability of laminar boundary-layer flow, or the minimum critical Reynolds number $R_{\theta_{cr_{min}}}$, is determined largely by the distribution of the product of the mean density and the mean vorticity across the boundary layer. An approximate estimate of $R_{\theta_{cr_{min}}}$ is obtained that serves as a criterion for

the influence of free-stream Mach number and thermal conditions at the solid surface on laminar stability. For zero pressure gradient, this estimate reads as follows:

$$R_{\theta_{cr_{min}}} \approx \frac{6}{T_1} \frac{\left[T\left(c_o\right)\right]^{1.76}}{c_o^{\,4}\sqrt{1 - M_o^{\,2}\left(1 - c_o\right)^2}}$$

where T is the ratio of temperature at a point within the boundary layer to free-stream temperature, T_1 is the ratio of temperature at the solid surface to the free-stream velocity, and c_o is the value of c (the ratio of phase velocity of disturbance to the free-stream velocity) for which $(1 - 2\lambda)v = 0.580$. The functions $v(c)$ and $\lambda(c)$ are defined as follows:

$$v(c) = \frac{-\pi\left(\frac{\partial w}{\partial \eta}\right)_1 c}{T_1}\left[\frac{T^2}{\left(\frac{\partial w}{\partial \eta}\right)^3}\frac{\partial}{\partial \eta}\left(\frac{1}{T}\frac{\partial w}{\partial \eta}\right)\right]_{w=c}$$

$$\lambda(c) = \frac{\eta\left(\frac{\partial w}{\partial \eta}\right)_1}{c} - 1$$

where

η nondimensional distance from surface

4. On the basis of the stability criterion in conclusion 3 and a study of the equations of mean motion, the effect of adding heat to the fluid through the solid surface is to reduce $R_{\theta_{cr_{min}}}$ and to

destabilize the flow, as compared with the flow over an insulated surface at the same Mach number. Withdrawing heat through the solid surface has exactly the opposite effect. The value of $R_{\theta_{cr_{min}}}$ for the laminar boundary-layer flow over an insulated surface decreases as the Mach number increases, and the flow is destabilized, as compared with the Blasius flow at low speeds.

5. When the free-stream velocity is supersonic, the laminar boundary layer is completely stabilized if the rate at which heat is withdrawn from the fluid through the solid surface reaches or exceeds a certain critical value. The critical rate of heat transfer, for which $R_{\theta_{cr_{min}}} \longrightarrow \infty$, is that which makes the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ sufficiently large negatively near the surface so that $\left[1 - 2\lambda(c)\right] v(c) = 0.580$ when $c = c_o = 1 - \frac{1}{M_o}$. Calculations for several supersonic Mach numbers between 1.30 and 5.00 show that for $M_o > 3$ (approx.) the critical rate of heat withdrawal for laminar stability is within the order of magnitude of the calculated rate of heat conduction through the solid surface that balances the heat radiated from the surface under free-flight conditions. Thus, for $M_o > 3$ (approx.) the laminar boundary-layer flow for thermal equilibrium is completely stable at all Reynolds numbers in the absence of a positive (adverse) pressure gradient in the direction of the free stream.

6. Detailed calculations of the curves of wave number (inverse wave length) against Reynolds number for the neutral boundary-layer disturbances for 10 representative cases of insulated and non-insulated surfaces show that also at subsonic speeds the quantitative effect on stability of the thermal conditions at the solid surface is very large. For example, at a Mach number of 0.70, the value of $R_{\theta_{cr_{min}}}$ is 63 when $T_1 = 1.25$ (heat added to fluid), $R_{\theta_{cr_{min}}} = 126$ when $T_1 = 1.10$ (insulated surface), and $R_{\theta_{cr_{min}}} = 5150$ when $T_1 = 0.70$ (heat withdrawn from fluid). Since $R_x* \approx 2.25R_\theta^2$, the effect on $R_{x*_{cr_{min}}}$ is even greater.

7. The results of the analysis of the stability of laminar boundary-layer flow by the linearized method of small perturbations must be applied with care to predictions of transition, which is a nonlinear phenomenon of a different order. Withdrawing heat from the

fluid through the solid surface, however, not only increases $R_{\theta_{cr_{min}}}$ but decreases the initial rate of amplification of the self-excited disturbances, which is roughly proportional to $1/\sqrt{R_{\theta_{cr_{min}}}}$; addition of heat to the fluid through the solid surface has the opposite effect. Thus, it can be concluded that (a) transition is delayed $\left(R_{\theta_{tr}}\right.$ increased$\left.\right)$ by withdrawing heat from the fluid and advanced by adding heat to the fluid through the solid surface, as compared with the insulated surface at the same Mach number, (b) for the insulated surface, transition occurs earlier as the Mach number is increased, (c) when the free stream velocity is supersonic, transition never occurs if the rate of heat withdrawal from the fluid through the solid surface reaches or exceeds the critical value for which $R_{\theta_{cr_{min}}} \longrightarrow \infty$. (See conclusion 5.)

Unlike laminar instability, transition to turbulent flow in the boundary layer is not a purely local phenomenon but depends on the previous history of the flow. The quantitative effect of thermal conditions at the solid surface on transition depends on the existing pressure gradient in the direction of the free stream, on the part of the solid surface to which heat is applied, and so forth, as well as on the initial magnitude of the disturbances (level of free-stream turbulence).

A comparison between conclusion 7(a), based on the results of the stability analysis, and experimental investigations of the effect of surface heating on transition at low speeds shows that the results of the present paper give the proper direction of this effect.

8. The results of the present study of laminar stability can be extended to include laminar boundary-layer flows of a gas in which there is a pressure gradient in the direction of the free stream. Although further study is required, it is presumed that only the local mean velocity-temperature distribution determines the stability of the local boundary-layer flow. If that should be the case, the effect of a pressure gradient on laminar stability could be easily calculated through its effect on the local distribution of the product of mean density and mean vorticity across the boundary layer.

When the free-stream velocity at the "edge" of the boundary layer is supersonic, by analogy with the stabilizing effect of a withdrawal of heat from the fluid, it is expected that the laminar boundary-layer flow is completely stable at all Reynolds numbers when the negative (favorable) pressure gradient reaches or exceeds a certain critical value that depends only on the Mach number and the properties of the gas. The laminar boundary-layer flow over a surface in thermal equilibrium should be completely stable for $M_o > M_s$, say, where $M_s < 3$ if there is a negative pressure gradient in the direction of the free stream.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., September 5, 1946

APPENDIX A

CALCULATION OF INTEGRALS APPEARING IN THE INVISCID SOLUTIONS

In order to calculate the limits of stability of the laminar boundary layer from relations (2.21) to (2.29) between the values of phase velocity, wave number, and Reynolds number, it is first necessary to calculate the values of the integrals K_1, H_1, H_2, N_2, M_3, N_3, and so forth, which appear in the expressions for the inviscid solutions $\varphi_1(y)$ and $\varphi_2(y)$ and their derivatives at the edge of the boundary layer. These integrals are as follows (equations (2.13), (2.9), and (2.10)):

$$H_1(c) = \int_{y_1}^{y_2} \frac{(w - c)^2}{T} dy$$

$$K_1(c) = \int_{y_1}^{y_2} \frac{T - M_o^2(w - c)^2}{(w - c)^2} dy$$

$$N_2(c) = K_1H_1 - K_2 = \int_{y_1}^{y_2} \frac{T - M_o^2(w - c)^2}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} dy = H_2(c)$$

$$M_3(c) = H_2H_1 - H_3 = \int_{y_1}^{y_2} \frac{(w - c)^2}{T} dy \int_{y}^{y_2} \frac{T - M_o^2(w - c)^2}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} dy$$

$$N_3(c) = K_1H_2 - K_3$$

$$= \int_{y_1}^{y_2} \frac{T - M_o^2(w - c)^2}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} dy \int_{y}^{y_2} \frac{T - M_o^2(w - c)^2}{(w - c)^2} dy$$

and so forth.

Terms of higher order than α^3 in the series expressions for φ_1 and φ_2 are neglected. When $\alpha < 1$, the error involved is small because the terms in the series decline like $\frac{\alpha^n}{n!}$. Even for $\alpha > 1$, however, this approximation is justified, at least for the values of c that appear in the stability calculations for the 10 representative cases selected in the present paper. For example, the leading term in R.P. $N_{2k+1}(c)$, where $k = 2,3, . . .,$ is approximately $\frac{1}{k!}\left[\frac{c^3}{3(1 - c)}\right]^{k-1}$ multiplied by the leading term in R.P. $N_3(c)$. The quantity in the brackets is at most 0.12 in the present calculations; for example, R.P. $N_5(c) \approx 0.06$ R.P. $N_3(c)$. Moreover, R.P. $N_{2k}(c) \approx (1 - c)$ R.P. $N_{2k+1}(c)$. Similar approximate relations exist between R.P. $M_{2k}(c)$ and R.P. $M_3(c)$; and, in addition, R.P. $M_3(c) \approx (1 - c)\,\frac{c^3}{6}$ R.P. $N_3(c) \approx 0.015$ R.P. $N_3(c)$, at most.

The only integral for which the imaginary part is calculated is $K_1(c)$. At the end of this appendix, it is shown that the contributions of the imaginary parts of H_2, M_3, and N_3 are negligible in comparison with the contribution of I.P. $K_1(c)$.

General Plan of Calculation

The method of calculation adopted must take into account the fact that the value of $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ at the point $y = y_c$, where $w = c$, strongly influences the stability of the laminar boundary layer. Accordingly, the integrals are broken into two parts; for example,

$$K_1(c) = \int_{y_1}^{y_j} \frac{T}{(w - c)^2}\,dy + \int_{y_j}^{y_2} \frac{T}{(w - c)^2}\,dy - M_o^2$$

$$= K_{11}(c) + K_{12}(c) - M_o^2$$

where $y_j > y_c$. The integral $K_{11}(c)$, which involves $\left[\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)\right]_{w=c}$, is calculated very accurately, whereas $K_{12}(c)$ is calculated by a more approximate method as follows:

$$K_{12}(c) = \int_{y_j}^{y_2} \frac{T}{(w - c)^2}\,dy \tag{1}$$

This integral is evaluated as a power series in c. The velocity profile $w(y)$ is approximated by a parabolic arc plus a straight-line segment for purposes of integration. In the more complex integrals H_2, M_3, and N_3, the indefinite integrals $\int_{y_j}^{y} \frac{T}{(w - c)^2}\,dy$ and $\int_{y}^{y_2} \frac{T}{(w - c)^2}\,dy$ are evaluated by 21 or 41 point numerical integration by means of Simpson's rule. The values of $w(y)$ are read from the velocity profiles of figures 1 and 2. The value of $y_j - y_1 = a$ is 0.40 in the present series of calculations; this value is chosen so that the point $y = y_j$ is never too close to the singularity at $y = y_c$. Take

$$K_{11}(c) = \int_{y_1}^{y_j} \frac{T}{(w - c)^2}\,dy \tag{2}$$

The integral $K_{11}(c)$, or the indefinite integral $\int_{y_1}^{y} \frac{T}{(w - c)^2} dy$ that appears in H_2, M_3, and N_3, is evaluated by expanding the integrand in a Taylor's series in $y - y_c$ and then integrating the series term by term. The path of integration must be taken below the point $y = y_c$ in the complex y-plane.

Instead of calculating the values of the velocity and temperature derivatives $w_c^{(n)}$ and $T_c^{(n)}$ directly, it is simpler to relate these derivatives to their values at the surface by Taylor's series of the form

$$w_c^{(n)} = w_1^{(n)} + w_1^{(n+1)} \left(y_c - y_1\right) + \frac{w_1^{(n+2)}}{2!} \left(y_c - y_1\right)^2 + \ldots$$

The derivatives at the surface $w_1^{(n)}$ and $T_1^{(n)}$ are calculated from the equations of mean motion (appendix B).

The integral $K_{11}(c)$, for example, is finally obtained as a power series in $y_c - y_1 = \sigma$ and in $y_j - y_c = a - \sigma$, plus terms involving $\log \sigma$. The phase velocity c is related to σ by

$$c = w_1' \left(\sigma + \frac{A_2}{2} \sigma^2 + \frac{A_3}{3!} \sigma^3 + \ldots\right)$$

where

$$A_k = \frac{w_1^{(k)}}{w_1'}$$

Terms up to the order of a^5 are retained in order to include all terms involving w_1^{vii}.

Detailed Calculations

In order to illustrate the method, the evaluation of $K_1(c)$ is given in some detail, as follows:

(1) Evaluation of $K_1(c)$:

$$K_1(c) = \int_{y_1}^{y_2} \frac{T}{(w - c)^2}\, dy - M_o^2$$

(a) Define

$$K_{11}(c) = \int_{y_1}^{y_3} \frac{T}{(w - c)^2}\, dy$$

Now

$$\frac{T}{(w - c)^2} = \frac{T}{\left(w_c'\right)^2 \left(y - y_c\right)^2 \psi^2}$$

where

$$\psi(y) = 1 + \frac{w_c''}{2w_c'}\left(y - y_c\right) + \frac{w_c'''}{3!w_c'}\left(y - y_c\right)^2 + \ldots$$

The function $\frac{T}{\psi^2}$ is developed in a Taylor's series around the point $w = c$ as follows:

$$\frac{T}{\psi^2} = \left(\frac{T}{\psi^2}\right)_{y=y_c} + \left(\frac{T}{\psi^2}\right)_c' \left(y - y_c\right) + \left(\frac{T}{\psi^2}\right)_c'' \frac{\left(y - y_c\right)^2}{2!} + \ldots$$

where

$$\psi_c = 1$$

$$\psi_c{}' = \frac{w_c{}''}{2w_c{}'}$$

$$\psi_c{}^{(k)} = \frac{w_c{}^{(k+1)}}{(k + 1)w_c{}'}$$

Then

$$K_{11}(c) = \frac{1}{(w_c{}')^2}\int_{y_1-y_c}^{y_j-y_c} \frac{d(y - y_c)}{(y - y_c)^2}\left[\left(\frac{T}{\psi^2}\right)_c + \left(\frac{T}{\psi^2}\right)_c{}'(y - y_c) + \left(\frac{T}{\psi^2}\right)_c{}''\frac{(y - y_c)^2}{2!} + \ldots\right]$$

and

$$K_{11}(c) = \frac{1}{(w_c{}')^2}\left\{\left[-\frac{T_c}{y - y_c}\right]_{y_1}^{y_j} + \left(\frac{T}{\psi^2}\right)_c{}' \ln\left(\frac{y_j - y_c}{y_1 - y_c}\right) + \frac{1}{2}\left(\frac{T}{\psi^2}\right)_c{}''(y_j - y_1)\right.$$

$$+ \frac{1}{12}\left(\frac{T}{\psi^2}\right)_c{}'''\left[(y_j - y_c)^2 - (y_1 - y_c)^2\right] + \ldots$$

$$\left. + \frac{1}{(k)(k + 1)!}\left(\frac{T}{\psi^2}\right)_c^{(k+1)}\left[(y_j - y_c)^k - (y_1 - y_c)^k\right] + \ldots\right\}$$

where

$$y_1 - y_c = \left|y_1 - y_c\right| e^{-i\pi}$$

$$y_j - y_c = \left(y_j - y_1\right) - \left(y_c - y_1\right) = a - \sigma$$

$$\sigma = y_c - y_1$$

The coefficients $\left(\frac{T}{\psi^2}\right)^{(k)}$ are expressed in terms of derivatives of T and w at $y = y_1$ as follows:

Define

$$f_k(y) = \frac{1}{(k - 1)\ k!}\ \frac{1}{(w')^2}\left(\frac{T}{\psi^2}\right)^k \qquad k \geqq 2$$

$$f_o(y) = -\frac{T}{(w')^2}$$

$$f_1(y) = \frac{1}{(w')^2}\left(\frac{T}{\psi^2}\right)' = -\frac{T^2}{(w')^3}\frac{d}{dy}\left(\frac{w'}{T}\right)$$

Then

$$f_k\left(y_c\right) = \frac{1}{\left(w_c'\right)^2(k - 1)k!}\left[\left(\frac{T}{\psi^2}\right)^k\right]_{y_c}$$

$$= f_k\left(y_1\right) + f_k'\left(y_1\right)\left(y_c - y_1\right) + \frac{f_k''\left(y_1\right)}{2!}\left(y_c - y_1\right)^2 + \ldots$$

(The method adopted for the calculation of $f_k^{(n)}\left(y_1\right)$ from the velocity and temperature derivatives $w_1^{(j)}$ and $T_1^{(j)}$ is given at the end of this appendix.)

From the expression for $K_{11}(c)$,

$$\begin{aligned}
\text{I.P. } K_{11}(c) &= \text{I.P. } K_1(c) \\
&= \pi f_1\left(y_c\right) \\
&= \pi\left[f_1\left(y_1\right) + \sigma f_1'\left(y_1\right) + \ldots + \frac{\sigma^5}{5!} f_1^{v}\left(y_1\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
\text{R.P. } K_{11}(c) + \frac{T_1}{w_1'c} = {}& c_o + c_1\sigma + c_2\sigma^2 + \ldots + c_5\sigma^5 \\
&+ \frac{\text{I.P. } K_{11}(c)}{\pi} \ln\left(\frac{a - \sigma}{\sigma}\right) + \frac{1}{a - \sigma}\left[f_o\left(y_1\right)\right. \\
&+ \sigma f_o'\left(y_1\right) + \ldots + \frac{\sigma^k f_o^{(k)}\left(y_1\right)}{k!} + \ldots \\
&\left.+ \frac{\sigma^6 f_o^{vi}\left(y_1\right)}{720}\right]
\end{aligned}$$

where

$$\sigma = y_c - y_1$$

$$c_k = s_k + \frac{f_o^{(k+1)}\left(y_1\right)}{(k + 1)!} - d_{k+1} f_o\left(y_1\right) \qquad 0 \leqq k \leqq 5 \qquad \left(s_5 = 0\right)$$

$$s_o = af_2\left(y_1\right) + a^2 f_3\left(y_1\right) + a^3 f_4\left(y_1\right) + a^4 f_5\left(y_1\right) + a^5 f_6\left(y_1\right) + \ldots$$

$$s_1 = af_2'(y_1) + a^2f_3'(y_1) + a^3f_4'(y_1) + a^4f_5'(y_1) + \ldots$$

$$- \left[2af_3(y_1) + 3a^2f_4(y_1) + 4a^3f_5(y_1) + 5a^4f_6(y_1) + \ldots\right]$$

$$s_2 = \frac{1}{2}\left[af_2''(y_1) + a^2f_3''(y_1) + a^3f_4''(y_1) + \ldots\right]$$

$$- \left[2af_3'(y_1) + 3a^2f_4'(y_1) + 4a^3f_5'(y_1) + \ldots\right]$$

$$+ \left[3af_4(y_1) + 6a^2f_5(y_1) + 10a^3f_6(y_1) + \ldots\right]$$

$$s_3 = \frac{1}{6}\left[af_2'''(y_1) + a^2f_3'''(y_1) + \ldots\right] - \frac{1}{2}\left[2af_3''(y_1) + 3a^2f_4''(y_1) + \ldots\right]$$

$$+ \left[3af_4'(y_1) + 6a^2f_5'(y_1) + \ldots\right] - \left[4af_5(y_1) + 10a^2f_6(y_1) + \ldots\right]$$

$$s_4 = \frac{1}{24}\left[af_2^{iv}(y_1) + \ldots\right] - \frac{1}{6}\left[2af_3'''(y_1) + \ldots\right] + \frac{1}{2}\left[3af_4''(y_1) + \ldots\right]$$

$$- \left[4af_5'(y_1) + \ldots\right] + \left[5af_6(y_1) + \ldots\right]$$

$$d_k = -\sum_{r=1}^{k} \frac{A_{r+1}}{(r+1)!} d_{k-r} \qquad d_o = 1.0$$

$$A_k = \frac{w_1^{(k)}}{w_1'} \qquad a = 0.40$$

(b) Define

$$K_{12}(c) = \int_{y_j}^{y_2} \frac{T}{(w - c)^2}\, dy$$

$$= \int_{0.40}^{1.0} \frac{T}{(w - c)^2}\, d(y - y_1)$$

$$= \sum_{k=0}^{\infty} a_k (k + 1) c^k$$

where

$$a_k = \int_{0.40}^{1.0} \frac{T}{w^{k+2}}\, d(y - y_1)$$

The velocity profile $w(y)$ is approximated by a parabolic arc in the interval $0.40 \leqq y - y_1 \leqq y_3 - y_1$ and by a straight line $\left(w = \text{Constant} = w(y_3)\right)$ in the interval $y_3 - y_1 \leqq y - y_1 \leqq 1.0$. The value of y_3 is determined by imposing the condition that the area under the parabolic-arc straight-line segment equals the area under the actual velocity profile $w(y)$ in the interval $0.40 \leqq y - y_1 \leqq 1.0$. The parabolic arc $w = l + m(y - y_1) + n(y - y_1)^2$ is determined by the following conditions:

when $y = y_4 < 1$,

$$w = 1$$

$$w' = 0$$

when $y = y_j$ and $y_j - y_1 = 0.40$,

$$w = w(y_j)$$

where $w(y_j)$ is read off the velocity profile of figures 1 and 2. The value of y_4 is chosen so that the parabolic arc fits the velocity curve $w(y)$ closely over the widest possible range.

For $\sigma = 1$,

$$T = T_1 - \left[\left(T_1 - 1\right) - \frac{\gamma - 1}{2} M_o^2\right] w - \frac{\gamma - 1}{2} M_o^2 w^2$$

Therefore

$$a_k = T_1\left(I_{k+2} + J_{k+2}\right) - \left[\left(T_1 - 1\right) - \frac{\gamma - 1}{2} M_o^2\right]\left(I_{k+1} + J_{k+1}\right) - \frac{\gamma - 1}{2} M_o^2\left(I_k + J_k\right)$$

where

$$I_k = \int_{0.40}^{y_3 - y_1} \frac{d(y - y_1)}{w^k}$$

and

$$J_k = \int_{y_3 - y_1}^{1.0} \frac{d(y - y_1)}{\left[w(y_3)\right]^k} = \frac{1 - (y_3 - y_1)}{\left[w(y_3)\right]^k}$$

I_k is evaluated by approximating $w(y)$ by a parabolic arc as follows:

$$I_1 = \frac{1}{\sqrt{A}} \ln \left[\frac{\sqrt{A} - m - 2n(y - y_1)}{\sqrt{A} + m + 2n(y - y_1)}\right]_{0.40}^{y_3 - y_1}$$

$$I_k = -\frac{1}{(k-1)A}\left[\frac{m + 2n(y - y_1)}{\left[l + m(y - y_1) + n(y - y_1)^2\right]^{k-1}}\right]_{0.40}^{y_3 - y_1} + \frac{2k-3}{2k-2}\,\frac{4(-n)}{A}\,I_{k-1}$$

where $A = m^2 - 4ln$.

As a control in the calculation of the series expression $\sum_{k=0}^{\infty} a_k(k+1)c^k$ for $K_{12}(c)$, use is made of the fact that, from the definition of I_k and J_k,

$$\lim_{k \longrightarrow \infty}\left(I_k + J_k\right) = \frac{1}{k\left[\dfrac{w'(y_j)}{w(y_j)}\right]\left[w(y_j)\right]^k}$$

and therefore

$$\lim_{k \longrightarrow \infty}\left(\frac{a_{k+1}}{a_k}\right) = \frac{1}{w(y_j)}\,\frac{k}{k+1}$$

The remainder after N - terms in the series for $K_{12}(c)$ is given approximately by

$$\frac{\left[(N + 1)\text{ term}\right]}{\left[1 - \frac{c}{w\left(y_j\right)}\right]}$$

The real part of $K_1(c)$ is obtained by combining the results of (a) and (b); that is,

$$\text{R.P. } K_1(c) = \text{R.P.}\left[K_{11}(c) + \frac{T_1}{w_1'c}\right] + K_{12}(c) - M_o^2$$

(2) Evaluation of $H_1(c)$:

$$H_1(c) = \int_{y_1}^{y_2} \frac{(w - c)^2}{T} \, dy$$

The integrand of this integral is free of singularities in the region of the complex y-plane bounded by $y = y_1$ and $y = y_2$; therefore $H_1(c)$ is evaluated by purely numerical integration. The actual procedure employed for the calculation of integrals of this type is as follows: (The integral $H_1(c)$ serves as an illustration.)

(a) Define

$$H_1(c) = \frac{1}{b}\left(\int_0^b \rho w^2 \, d\eta - 2c \int_0^b \rho w \, d\eta + c^2 \int_0^b \rho \, d\eta\right)$$

where

$$\eta = y^* \sqrt{\frac{\overline{u_o^*}}{\overline{\nu_o^*} \; x^*}}$$

and

$$b = \delta\sqrt{\frac{\overline{u_o}^*}{\overline{\upsilon_o}^*\ x^*}}$$

(b) With the approximation that the viscosity varies linearly with the absolute temperature, the velocity w is the same function of the nondimensional stream function ζ as in the Blasius flow; that is,

$$w = w(\zeta) = w_B(\zeta)$$

where ζ is defined by the relation $d\zeta = \rho w\ d\eta$ (appendix B).

From these relations

$$\rho w^n\ d\eta = \left[w(\zeta)\right]^{n-1} d\zeta = \left[w_B(\eta_B)\right]^n d\eta_B$$

since $d\zeta = w_B\ d\eta_B$. Moreover,

$$d\eta = \frac{d\zeta}{\rho w(\zeta)} = T(w_B)\ d\eta_B$$

where

$$T(w_B) = T_1 - \left[(T_1 - 1) - \frac{\gamma - 1}{2} M_o^2\right] w_B - \frac{\gamma - 1}{2} M_o^2 w_B^2$$

for $\sigma = 1$.

(c) Finally, from the relations given in (b),

$$H_1(c) = \frac{1}{b}\left(\int_0^{b_o} w_B{}^2\, d\eta_B - 2c\int_0^{b_o} w_B\, d\eta_B + c^2\right)$$

where b_o is the value of $\delta\sqrt{\frac{\overline{u_o^*}}{\overline{\upsilon_o^*}\ x^*}}$ for the Blasius flow. For the insulated surfaces, b_o, which is somewhat arbitrary, was chosen as 5.60; whereas for the noninsulated surfaces, $b_o = 6.00$. (The value of w_B at $\eta_B = 5.60$ is 0.9950; when $\eta_B = 6.00$, $w_B = 0.9975$. The value of b for the insulated surfaces is the value of η at which $w = 0.9950$; whereas b for the noninsulated surfaces is the value of η for which $w = 0.9975$.) The advantage of this procedure is that the integrals $\int_0^{b_o} w_B{}^n\, d\eta_B$ are calculated once and for all and the value of $H_1(c)$ depends only upon the values of b and c. In fact,

$$\int_0^{b_o} w_B{}^2\, d\eta_B = b_o - \left(\delta^*\sqrt{\frac{\overline{u_o^*}}{\overline{\upsilon_o^*}\ x^*}}\right)_B - \left(\theta\sqrt{\frac{\overline{u_o^*}}{\overline{\upsilon_o^*}\ x^*}}\right)_B = b_o - 2.3967$$

since

$$\left(\delta^*\sqrt{\frac{\overline{u_o^*}}{\overline{\upsilon_o^*}\ x^*}}\right)_B = 1.730$$

and

$$\left(\theta\sqrt{\frac{\overline{u_o^*}}{\overline{\upsilon_o^*}\ x^*}}\right)_B = 0.6667$$

Also,

$$\int_0^{b_o} w_B \, d\eta_B = b_o - 1.730$$

and

$$b = \int_0^b d\eta = \int_0^{b_o} T \, d\eta_B$$

$$= b_o + 1.73\left(T_1 - 1\right) + 0.6667 \frac{\gamma - 1}{2} M_o^2$$

$$= b_o + 1.73 \left[\left(T_1 - 1\right) - \frac{\gamma - 1}{2} M_o^2\right] + 2.3967 \frac{\gamma - 1}{2} M_o^2$$

See appendix B. (Incidentally, the last relation shows the effect of free-stream Mach number and thermal conditions at the solid surface on the "thickness" of the boundary layer.)

(3) Evaluation of $H_2(c)$:

$$H_2(c) = \int_{y_1}^{y_2} \frac{T - M_o^2(w - c)^2}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} dy$$

$$= \int_{y_1}^{y_2} \frac{T}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} dy - M_o^2 \int_{y_1}^{y_2} \int_{y_1}^{y} \frac{(w - c)^2}{T} dy \, dy$$

Define

$$H_{2_1}(c) = \int_{y_1}^{y_2} \frac{T}{(w - c)^2}\, dy \int_{y_1}^{y} \frac{(w - c)^2}{T}\, dy$$

$$H_{2_2}(c) = \int_{y_1}^{y_2} \int_{y_1}^{y} \frac{(w - c)^2}{T}\, dy\, dy$$

(a) The integral $H_{2_2}(c)$ is evaluated by methods similar to those already outlined for the evaluation of $H_1(c)$. Thus

$$H_{2_2}(c) = \int_{y_1}^{y_2} \int_{y_1}^{y} \frac{(w - c)^2}{T}\, dy\, dy$$

$$= \int_{y_1}^{y_2} dy \int_{y_1}^{y} \rho w^2\, dy - 2c \int_{y_1}^{y_2} dy \int_{y_1}^{y} \rho w\, dy + c^2 \int_{y_1}^{y_2} dy \int_{y_1}^{y} \rho\, dy$$

$$= \frac{1}{b^2}\left(\int_0^{b_o} T\, d\eta_B \int_0^{\eta_B} {w_B}^2\, d\eta_B - 2c \int_0^{b_o} T\, d\eta_B \int_0^{\eta_B} w_B\, d\eta_B + c^2 \int_0^{b_o} T\eta_B\, d\eta_B\right)$$

where

$$T = T_1 - \left[\left(T_1 - 1\right) - \frac{\gamma - 1}{2} {M_o}^2\right] w_B - \frac{\gamma - 1}{2} {M_o}^2 {w_B}^2$$

The nine integrals in the expression for $H_{2_2}(c)$ are evaluated by numerical integration using Simpson's rule.

(b) Define

$$H_{2_1}(c) = \int_{y_1}^{y_2} \frac{T}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} dy$$

$$= \int_{y_1}^{y_j} \frac{T}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} dy + \int_{y_j}^{y_2} \frac{T}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} dy$$

Define

$$H_{2_{11}}(c) = \int_{y_1}^{y_j} \frac{T}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} dy$$

$$H_{2_{12}}(c) = \int_{y_j}^{y_2} \frac{T}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} dy$$

The integral $H_{2_{12}}(c)$ is evaluated as follows:

$$H_{2_{12}}(c) = \int_{y_j}^{y_2} \frac{T}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} dy$$

$$= \int_{y_j}^{y_2} \frac{T}{(w - c)^2} dy \int_{y_1}^{y_2} \frac{(w - c)^2}{T} dy - \int_{y_j}^{y_2} \frac{T}{(w - c)^2} dy \int_{y}^{y_2} \frac{(w - c)^2}{T} dy$$

But

$$\int_{y_j}^{y_2} \frac{T}{(w - c)^2}\, dy = K_{12}(c)$$

and

$$\int_{y_1}^{y_2} \frac{(w - c)^2}{T}\, dy = H_1(c)$$

so that

$$H_{2_{12}}(c) = K_{12}(c)\, H_1(c) - \int_{y_j}^{y_2} \frac{T}{(w - c)^2}\, dy \int_{y}^{y_2} \frac{(w - c)^2}{T}\, dy$$

Define

$$P(c) = \int_{y_j}^{y_2} \frac{T}{(w - c)^2}\, G(y;\ c)\, dy$$

$$= \frac{1}{b^2} \int_{0.4b}^{b} \frac{T}{(w - c)^2}\, G(\eta;\ c)\, d\eta$$

where

$$G(\eta;\ c) = \int_{\eta}^{b} \frac{(w - c)^2}{T}\, d\eta$$

$$= \int_{\eta}^{b} \frac{w^2}{T}\, d\eta - 2c \int_{\eta}^{b} \frac{w}{T}\, d\eta + c^2 \int_{\eta}^{b} \frac{d\eta}{T}$$

and

$$T = T_1 - \left[\left(T_1 - 1\right) - \frac{\gamma - 1}{2} M_o^2\right] w - \frac{\gamma - 1}{2} M_o^2 w^2$$

The integral $P(c)$ is evaluated by numerical integration using Simpson's rule; the required values of w are read directly off the velocity profiles of figures 1 and 2. Finally,

$$H_{2_{12}}(c) = K_{12}(c)\ H_1(c) - P(c)$$

The integral $H_{2_{11}}(c)$ is evaluated in exactly the same way as $K_{11}(c)$ where

$$\frac{(w - c)^2}{T} = \left(w_c'\right)^2 \left(y - y_c\right)^2 \left(\frac{\psi^2}{T}\right)$$

and

$$\psi(y) = 1 + \frac{w_c''}{2w_c'} \left(y - y_c\right) + \frac{w_c'''}{3!\,w_c'} \left(y - y_c\right)^2 + \ldots$$

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 1360

THE STABILITY OF THE LAMINAR BOUNDARY LAYER IN A COMPRESSIBLE FLUID

By Lester Lees

Langley Memorial Aeronautical Laboratory
Langley Field, Va.

Washington
July 1947

TABLE OF CONTENTS

Page

Summary 1

Introduction 3

Symbols 3

1. Preliminary Considerations 6

2. Calculation of the Limits of Stability of the Laminar Boundary Layer in a Viscous Conductive Gas 11

3. Destabilizing Influence of Viscosity at Very Large Reynolds Numbers; Extension of Heisenberg's Criterion to the Compressible Fluid 21

4. Stability of Laminar Boundary Layer at Large Reynolds Numbers 27

a. Subsonic Free-Stream Velocity $\left(M_o < 1\right)$ 27

b. Supersonic Free-Stream Velocity $\left(M_o > 1\right)$ 32

5. Criterion for the Minimum Critical Reynolds Number 39

6. Physical Significance of Results of Stability Analysis 45

a. General 45

b. Effect of Free-Stream Mach Number and Thermal Conditions at Solid Surface on Stability of Laminar Boundary Layer 46

c. Results of Detailed Stability Calculations for Insulated and Noninsulated Surfaces 52

d. Instability of Laminar Boundary Layer and Transition to Turbulent Flow 53

7. Stability of the Laminar Boundary-Layer Flow of a Gas with a Pressure Gradient in the Direction of the Free Stream 56

Conclusions 59

Appendixes 64

Appendix A - Calculation of Integrals Appearing in the Inviscid Solutions 64

General Plan of Calculation 65

Detailed Calculations 68

Evaluation of $f_k^{(m)}$ 105

Order of Magnitude of Imaginary Parts of Integrals H_2, M_3, and N_3 110

Appendix B - Calculation of Mean-Velocity and Mean-Temperature Distribution across Boundary Layer and the Velocity and Temperature Derivatives at the Solid Surface 113

Mean Velocity-Temperature Distribution across Boundary Layer 113

Calculation of Mean-Velocity and Mean-Temperature Derivatives 119

Appendix C - Rapid Approximation to the Function $(1 - 2\lambda)v(c)$ and the Minimum Critical Reynolds Number 125

Appendix D - Behavior of $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ from Equations of Mean Motion 129

Appendix E - Calculation of Critical Mach Number for Stabilization of Laminar Boundary Layer 132

References 135

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE NO. 1360

THE STABILITY OF THE LAMINAR BOUNDARY LAYER IN A COMPRESSIBLE FLUID

By Lester Lees

SUMMARY

The present paper is a continuation of a theoretical investigation of the stability of the laminar boundary layer in a compressible fluid. An approximate estimate for the minimum critical Reynolds number $R_{\theta_{cr_{min}}}$, or stability limit, is obtained in terms of the distribution of the kinematic viscosity and the product of the mean density $\overline{\rho^*}$ and mean vorticity $\frac{d\overline{u^*}}{dy^*}$ across the boundary layer. With the help of this estimate for $R_{\theta_{cr_{min}}}$, it is shown that withdrawing heat from the fluid through the solid surface increases $R_{\theta_{cr_{min}}}$ and stabilizes the flow, as compared with the flow over an insulated surface at the same Mach number. Conduction of heat to the fluid through the solid surface has exactly the opposite effect. The value of $R_{\theta_{cr_{min}}}$ for the insulated surface decreases as the Mach number increases for the case of a uniform free-stream velocity. These general conclusions are supplemented by detailed calculations of the curves of wave number (inverse wave length) against Reynolds number for the neutral disturbances for 10 representative cases of insulated and noninsulated surfaces.

So far as laminar stability is concerned, an important difference exists between the case of a subsonic and supersonic free-stream velocity outside the boundary layer. The neutral boundary-layer disturbances that are significant for laminar stability die out exponentially with distance from the solid surface; therefore the phase velocity c^* of these disturbances is subsonic relative to the free-stream velocity $\overline{u_o^*}$ - or $\overline{u_o^*} - c^* < \overline{a_o^*}$, where $\overline{a_o^*}$ is the local sonic velocity. When $\frac{\overline{u_o^*}}{\overline{a_o^*}} = M_o < 1$ (where M_o is free-stream Mach number), it follows that $0 \leqq c^* \leqq c^*_{max}$; and any

laminar boundary-layer flow is ultimately unstable at sufficiently high Reynolds numbers because of the destabilizing action of viscosity near the solid surface, as explained by Prandtl for the incompressible fluid. When $M_o > 1$, however, $\frac{c^*}{u_o^*} > 1 - \frac{1}{M_o} > 0$. If the quantity $\left[\frac{d}{dy^*}\left(\overline{\rho^*}\,\frac{d\overline{u^*}}{dy^*}\right)\right]_{\overline{u^*}=c^*}$ is large enough negatively, the rate at which energy passes from the disturbance to the mean flow, which is proportional to $-c^*\left[\frac{d}{dy^*}\left(\overline{\rho^*}\,\frac{d\overline{u^*}}{dy^*}\right)\right]_{\overline{u^*}=c^*}$, can always be large enough to counterbalance the rate at which energy passes from the mean flow to the disturbance because of the destabilizing action of viscosity near the solid surface. In that case only damped disturbances exist and the laminar boundary layer is completely stable at all Reynolds numbers. This condition occurs when the rate at which heat is withdrawn from the fluid through the solid surface reaches or exceeds a critical value that depends only on the Mach number and the properties of the gas. Calculations show that for $M_o > 3$ (approx.) the laminar boundary-layer flow for thermal equilibrium - where the heat conduction through the solid surface balances the heat radiated from the surface - is completely stable at all Reynolds numbers under free-flight conditions if the free-stream velocity is uniform.

The results of the analysis of the stability of the laminar boundary layer must be applied with care to discussions of transition; however, withdrawing heat from the fluid through the solid surface, for example, not only increases $R_{\theta_{cr_{min}}}$ but also decreases the initial rate of amplification of the self-excited disturbances, which is roughly proportional to $1/\sqrt{R_{\theta_{cr_{min}}}}$. Thus, the effect of the thermal conditions at the solid surface on the transition Reynolds number $R_{\theta_{tr}}$ is similar to the effect on $R_{\theta_{cr_{min}}}$. A comparison between this conclusion and experimental investigations of the effect of surface heating on transition at low speeds shows that the results of the present paper give the proper direction of this effect.

The extension of the results of the stability analysis to laminar boundary-layer gas flows with a pressure gradient in the direction of the free stream is discussed.

INTRODUCTION

By the theoretical studies of Heisenberg, Tollmien, Schlichting, and Lin (references 1 to 5) and the careful experimental investigations of Liepmann (reference 6) and H. L. Dryden and his associates (reference 7), it has been definitely established that the flow in the laminar boundary layer of a viscous homogeneous incompressible fluid is unstable above a certain characteristic critical Reynolds number. When the level of the disturbances in the free stream is low, as in most cases of technical interest, this inherent instability of the laminar motion at sufficiently high Reynolds numbers is responsible for the ultimate transition to turbulent flow in the boundary layer. The steady laminar boundary-layer flow would always represent a possible solution of the steady equations of motion, but this steady flow is in a state of unstable dynamic equilibrium above the critical Reynolds number. Self-excited disturbances (Tollmien waves) appear in the flow, and these disturbances grow large enough eventually to destroy the laminar motion.

The question naturally arises as to how the phenomena of laminar instability and transition to turbulent flow are modified when the fluid velocities and temperature variations in the boundary layer are large enough so that the compressibility and conductivity of the fluid can no longer be neglected. The present paper represents the second phase of a theoretical investigation of the stability of the laminar boundary-layer flow of a gas, in which the compressibility and heat conductivity of the gas as well as its viscosity, are taken into account. The first part of this work was presented in reference 8. The objects of this investigation are (1) to determine how the stability of the laminar boundary layer is affected by the free-stream Mach number and the thermal conditions at the solid boundary and (2) to obtain a better understanding of the physical basis for the instability of laminar gas flows. In this sense, the present study is an extension of the Tollmien-Schlichting analysis of the stability of the laminar flow of an incompressible fluid, but the investigation is also concerned with the general question of boundary-layer disturbances in a compressible fluid and their possible interactions with the main external flow.

SYMBOLS

With minor exceptions the symbols used in this paper are the same as those introduced in reference 8. Physical quantities are denoted by an asterisk, or star, whereas the corresponding non-dimensional quantities are unstarred. A bar over a quantity denotes mean value; a prime denotes a fluctuation; the subscript o denotes free-stream values at the "edge" of the boundary layer; the subscript 1 denotes values at the solid surface; and the subscript c denotes values at the inner "critical layer", where the phase velocity of the disturbance equals the mean flow velocity. The free-stream values are the characteristic measures for all non-dimensional quantities. The characteristic length measure is the boundary-layer thickness δ, except where otherwise indicated. Note that in order to conform with standard notation, the symbol δ for boundary-layer thickness is unstarred, whereas the symbols δ^* and θ are used for boundary-layer displacement thickness and boundary-layer momentum thickness, respectively.

x^* distance along surface

y^* distance normal to surface

t^* time

u^* component of velocity in x^*-direction

$w = \dfrac{\overline{u^*}}{\overline{u_o^*}}$

v^* component of velocity in y^*-direction

$\varphi = \dfrac{v^{*\prime}}{\overline{u_o^*}}$

ψ^* stream function for mean flow

ρ^* density of gas

p^* pressure of gas

T^* temperature of gas

τ^* laminar shear stress

μ_1^* ordinary coefficient of viscosity of gas

ν^* kinematic viscosity of gas $\left(\mu_1^*/\rho^*\right)$

k^*	thermal conductivity of gas
c_v	specific heat at constant volume
c_p	specific heat at constant pressure
R^*	gas constant per gram
γ	ratio of specific heats (c_p/c_v); 1.405 for air
c^*	complex phase velocity of boundary-layer disturbance
λ^*	wave length of boundary-layer disturbance
δ	boundary-layer thickness
δ^*	boundary-layer displacement thickness $\left(\int_0^\infty (1 - \rho w)\,dy^*\right)$
θ	boundary-layer momentum thickness $\left(\int_0^\infty \rho w(1 - w)\,dy^*\right)$
α^*	wave number of boundary-layer disturbance $(2\pi/\lambda^*)$

$$\alpha = \frac{2\pi}{\lambda^*/\delta}$$

$$\alpha_\theta = \frac{2\pi}{\lambda^*/\theta}$$

R	Reynolds number $\left(\dfrac{\overline{\rho_o^*}\ \overline{u_o^*}\ \delta}{\overline{\mu_{1_o}^*}}\right)$

$$R_\theta = \frac{\overline{\rho_o^*}\ \overline{u_o^*}\ \theta}{\overline{\mu_{1_o}^*}}$$

M_o	Mach number $\left(\dfrac{\overline{u_o^*}}{\sqrt{\gamma R^*\ \overline{T_o^*}}}\right)$

σ Prandtl number $\left(c_p \frac{\overline{\mu_{1_o}^*}}{\overline{k_o^*}}\right)$

1. PRELIMINARY CONSIDERATIONS

In the first phase of this investigation (reference 8) the stability of the laminar boundary-layer flow of a gas is analyzed by the method of small perturbations, which was already so successfully utilized for the study of the stability of the laminar flow of an incompressible fluid. (See reference 5.) By this method a nonsteady gas flow is investigated in which all physical quantities differ from their values in a given steady gas flow by small perturbations that are functions of the time and the space coordinates. This nonsteady flow must satisfy the complete gas-dynamic equations of motion and the same boundary conditions as the given steady flow. The question is whether the nonsteady flow damps to the steady flow, oscillates about it, or diverges from it with time - that is, whether the small perturbations are damped, neutral, or self-excited disturbances in time, and thus whether the given steady gas flow is stable or unstable. The analysis is particularly concerned with the conditions for the existence of neutral disturbances, which mark the transition from stable to unstable flow and define the minimum critical Reynolds number.

In order to bring out some of the principal features of the stability problem without becoming involved in hopeless mathematical complications, the solid boundary is taken as two dimensional and of negligible curvature and the boundary-layer flow is regarded as plane and essentially parallel; that is, the velocity component in the direction normal to the surface is negligible and the velocity component parallel to the surface is a function mainly of the distance normal to the surface. The small disturbances, which are also two dimensional, are analyzed into Fourier components, or normal modes, periodic in the direction of the free stream; and the amplitude of each one of these partial oscillations is a function of the distance normal to the solid surface, that is, $u^{*\prime} = \overline{u_o^*}\, f(y)\, e^{i\alpha(x-ct)}$.

In the study of the stability of the laminar boundary layer, it will be seen that only the local properties of the "parallel" flow are significant. To include the variation of the mean velocity in the direction of the free stream or the velocity component normal to the solid boundary in the problem would lead only to higher order terms in the differential equations governing the disturbances, since both of these factors are inversely proportional to the local Reynolds number based on the boundary-layer thickness. (See, for example, reference 2.) By a careful analysis, Pretsch has shown that even with a pressure gradient in the direction of the free stream the local mean-velocity distribution alone determines the stability characteristics of the local boundary-layer flow at large Reynolds numbers (reference 9). Such a statement applies only to the stability of the flow <u>within</u> the boundary layer. For the interaction between the boundary layer and a main "external" supersonic flow, for example, it is obviously the variation in boundary-layer thickness and mean velocity along the surface that is significant. (See reference 10.)

The aforementioned considerations also lead quite naturally to the study of individual partial oscillations of the form $f(y)\ e^{i\alpha(x-ct)}$, for which the differential equations of disturbance do not contain x and t explicitly. These partial oscillations are ideally suited for the study of instability, for in order to show that a flow is unstable it is unnecessary to consider the most general possible disturbance; in fact, the simplest will suffice. It is only necessary to show that a particular disturbance satisfying the equations of motion and the boundary conditions is self-excited or, in this case, that the imaginary part of the complex phase velocity c is positive.

In reference 8 the differential equations governing one normal mode of the disturbances in the laminar boundary layer of a gas were derived and studied very thoroughly. The complete set of solutions of the disturbance equations was obtained and the physical boundary conditions that these solutions satisfy were investigated. It was found that the final relation between the values of c, α, and R that determines the possible neutral disturbances (limits of stability) is of the <u>same form</u> in the compressible fluid as in the incompressible fluid, to a first approximation. The basis for this result is the fact that for Reynolds numbers of the order of those encountered in most aerodynamic problems, the temperature disturbances have only a negligible effect on those particular velocity solutions of the disturbance equations that depend primarily on the viscosity (viscous solutions). To a first approximation, these viscous solutions therefore do not depend directly on the heat conductivity and are of the same form as in the incompressible fluid, except that they involve the Reynolds number based on the kinematic viscosity near the solid boundary (where the viscous forces are important) rather than in the free stream. In this first approximation, the second viscosity coefficient, which is a measure of the dependence of the pressure on the rate of change of density, does not affect the stability of the laminar boundary layer. From these results it was inferred that at large Reynolds numbers the influence of the viscous forces on the stability is essentially the same as in an incompressible fluid. This inference is borne out by the results of the present paper.

The influence of the inertial forces on the stability of the laminar boundary layer is reflected in the behavior of the asymptotic inviscid solutions of the disturbance equations, which are independent of Reynolds number in first approximation. The results obtained in reference 8 show that the behavior of the inertial forces is dominated by the distribution of the product of the mean density and mean vorticity $\rho\frac{dw}{dy}$ across the boundary layer. (The gradient of this quantity, or $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$, which plays the same role as the gradient of the vorticity in the case of an incompressible fluid, is a measure of the rate at which the x-momentum of the thin layer of fluid near the critical layer (where $w = c$) increases, or decreases, because of the transport of momentum by the disturbance.) In order to clarify the behavior of the inertial forces, the limiting case of an inviscid fluid $(R \rightarrow \infty)$ is studied in detail in reference 8. The following general criterions are obtained: (1) If the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ vanishes for some value of $w > 1 - \frac{1}{M_o}$, then neutral and self-excited subsonic disturbances exist and the inviscid compressible flow is unstable. (2) If the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ does not vanish for some value of $w > 1 - \frac{1}{M_o}$, then all subsonic disturbances of finite wave length are damped and the inviscid compressible flow is stable. (Outside the boundary layer, the relative velocity between the mean flow and the x-component of the phase velocity of a subsonic disturbance is less than the mean sonic velocity. The magnitude of such a disturbance dies out exponentially with distance from the solid surface.) (3) In general, a disturbance gains energy from the mean flow if $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ is positive at the critical layer (where $w = c$) and loses energy to the mean flow if $\left[\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)\right]_{w=c} < 0$.

The general stability criterions for inviscid compressible flow give some insight into the effect of the inertial forces on the stability, but they cannot be taken over bodily to the real compressible fluid. Of course, if a flow is unstable in the limiting case of an infinite Reynolds number, the flow is unstable for a certain finite range of Reynolds number. A compressible flow that is stable when $R \longrightarrow \infty$, however, is not necessarily stable at all finite Reynolds numbers when the effect of viscosity is taken into account. One of the objects of the present paper is to settle this question.

On the basis of the stability criterions obtained in reference 8, some general statements were made concerning the effect of thermal conditions at the solid boundary on the stability of laminar boundary-layer flow. It is concluded from physical reasoning and a study of the equations of mean motion that the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ vanishes for some value of $w > 0$ if $\left(\frac{\partial T}{\partial y}\right)_1 \leqq 0$, that is, if heat is <u>added</u> to the fluid through the solid surface or if the surface is insulated. If $\left(\frac{\partial T}{\partial y}\right)_1 > 0$ and is sufficiently large, that is, if heat is <u>withdrawn</u> from the fluid through the solid surface at a sufficient rate, the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ never vanishes. Thus, when $\left(\frac{\partial T}{\partial y}\right)_1 \leqq 0$, the laminar boundary-layer flow is destabilized by the action of the inertial forces but stabilized through the increase of kinematic viscosity near the solid surface. When $\left(\frac{\partial T}{\partial y}\right)_1 > 0$, the reverse is true. The question of which of these effects is predominant can be answered only by further study of the stability problem in a real compressible fluid.

In the present paper this investigation is continued along the following lines:

(1) A study is made of how the general criterions for instability in an inviscid compressible fluid are modified by the introduction of a small viscosity (stability at very large Reynolds numbers).

(2) The conditions for the existence of neutral disturbances at large Reynolds number are examined (study of asymptotic form of relation between eigen-values of c, α, and R).

(3) A relatively simple expression for the approximate value of the minimum critical Reynolds number is derived; this expression involves the local distribution of mean velocity and mean temperature across the boundary layer. This approximation will serve as a criterion from which the effect of the free-stream Mach number and thermal conditions at the solid surface on the stability of laminar boundary-layer flow is readily evaluated. The question of the relative influence of the kinematic viscosity and the distribution of $\rho\frac{dw}{dy}$ on stability would then be settled.

(4) The energy balance for small disturbances in the real compressible fluid is considered in an attempt to clarify the physical basis for the instability of laminar gas flows.

(5) In order to supplement the investigations outlined in the four preceding paragraphs, detailed calculations are made of the limits of stability, or the curve of α against R for the neutral disturbances for several representative cases of insulated and noninsulated surfaces. The results of the calculations are presented in figures 1 to 8 and tables I to IV. The method of computation of the stability limits is briefly outlined in reference 8, although the calculations were not carried out in that paper.

In the present investigation the work of Heisenberg (reference 1) and Lin (reference 5) on the stability of a real incompressible fluid is naturally an indispensable guide. In fact, the methods utilized in the present study are analogous to those developed for an incompressible fluid.

The present paper is concerned only with the subsonic disturbances. The amplitude of the subsonic disturbance dies out rapidly with distance from the solid boundary. In other words, the neutral subsonic disturbance is an "eigen-oscillation" confined mainly to the boundary layer and exists only for discrete eigen-values of c, α, and R that determine the limits of stability of laminar boundary-layer flow. Disturbances classified in reference 8 as neutral "supersonic," that is, disturbances such that the relative velocity between the x-component of the phase velocity of such a disturbance and the free-stream velocity is greater than the local mean sound speed in the free stream, are actually progressive sound waves that impinge obliquely on the boundary layer and are reflected with change of amplitude. For disturbances of this type the wave length and phase velocity are obviously completely arbitrary (eigenvalues are continuous), and these disturbances have no significance for boundary-layer stability.

When the free-stream velocity is supersonic $\left(M_o > 1\right)$, the subsonic boundary-layer disturbances must satisfy the requirement that $\overline{u_o^*} - c^* < \overline{a_o^*}$ or $c > 1 - \frac{1}{M_o}$ (for $M_o < 1$, $c \geq 0$). Now, by analogy with the case of an incompressible fluid it is to be expected that for values of c greater than some critical value of c_o, say, all subsonic disturbances are damped. Thus, when $M_o > 1$, there is the possibility that for certain mean velocity-temperature distributions across the boundary layer, neutral or self-excited disturbances satisfying the differential equations of motion, the boundary conditions, and, also, the physical requirement that $c > 1 - \frac{1}{M_o}$ cannot be found. In that event, the laminar boundary flow is stable at all Reynolds numbers. This interesting possibility is investigated in the present paper.

2. CALCULATION OF THE LIMITS OF STABILITY OF THE LAMINAR BOUNDARY LAYER IN A VISCOUS CONDUCTIVE GAS

In order that the complete system of solutions of the differential equations for the propagation of small disturbances in the laminar boundary layer shall satisfy the physical boundary conditions, the phase velocity must depend on the wave length, the Reynolds number, and the Mach number in a manner that is determined entirely by the local distribution of mean velocity and mean temperature across the boundary layer. In other words, the only possible subsonic disturbances in the laminar boundary layer are those for which there exists a definite relation of the form (reference 8)

$$c = c\left(\alpha, R, M_o^2\right) \tag{2.1}$$

Since α, R, and M_o^2 are real quantities, the relation (2.1) is equivalent to the two relations

$$c_r = c_r\left(\alpha,\ R,\ M_o^2\right) \tag{2.1a}$$

$$c_i = c_i\left(\alpha,\ R,\ M_o^2\right) \tag{2.1b}$$

The curve $c_i\left(\alpha, R, M_o^2\right) = 0$ $\left(\text{or}\ \ \alpha = \alpha\left(R, M_o^2\right)\right)$ for the neutral disturbances gives the limits of stability of the laminar boundary layer at a given value of the Mach number. From this curve can be determined the value of the Reynolds number below which disturbances of all wave lengths are damped and above which self-excited disturbances of certain wave lengths appear in a given laminar boundary-layer flow.

In reference 8, it is shown that the relation (2.1) between the phase velocity and the wave length takes the following form:

$$E\left(\alpha,\ c,\ M_o^2\right) = F(z) \tag{2.2}$$

In equation (2.2), $F(z)$ is the Tietjens function (reference 11) defined by the relation

$$F(z) = 1 + \frac{\displaystyle\int_{\infty}^{-z} \zeta^{3/2} H_{1/3}^{(1)} \left\{\frac{2}{3}(i\zeta)^{3/2}\right\} d\zeta}{\displaystyle z\int_{\infty}^{-z} \zeta^{1/2} H_{1/3}^{(1)} \left\{\frac{2}{3}(i\zeta)^{3/2}\right\} d\zeta} \tag{2.3}$$

where

$$z = \left(\frac{\alpha R w_c'}{\nu_c}\right)^{1/3} \left(y_c - y_1\right) \tag{2.4}$$

and the quantity $H_{1/3}^{(1)}$ is the Hankel function of the first kind of order $1/3$. The prime denotes differentiation with respect to y. The function $E\left(\alpha, c, M_o^2\right)$, which depends only on the asymptotic inviscid solutions φ_1 and φ_2 (section 4 of reference 8) and not on the Reynolds number, is defined as follows:

$$\left(y_1 - y_c\right) E\left(\alpha, c, M_o^2\right) = \frac{\begin{vmatrix} \varphi_{11} & \varphi_{12}' + \beta\varphi_{12} \\ \varphi_{21} & \varphi_{22}' + \beta\varphi_{22} \end{vmatrix}}{\begin{vmatrix} \dfrac{T_1\varphi_{11}' + M_o^2 w_1' c\varphi_{11}}{T_1 - M_o^2 c^2} & \varphi_{12}' + \beta\varphi_{12} \\ \dfrac{T_1\varphi_{21}' + M_o^2 w_1' c\varphi_{21}}{T_1 - M_o^2 c^2} & \varphi_{22}' + \beta\varphi_{22} \end{vmatrix}} \tag{2.5}$$

where

$$\left.\begin{aligned} \beta &= \alpha\sqrt{1 - M_o^2(1 - c)^2} \\ \varphi_{ij} &= \varphi_i\left(y_j\right) \\ i, j &= 1, 2 \end{aligned}\right\} \tag{2.6}$$

and y_1 and y_2 are the coordinates of the solid surface and the "edge" of the boundary layer, respectively.

The Tietjens function was carefully recalculated in reference 8, and the real and imaginary parts of the function $\Phi(z) = \dfrac{1}{1 - F(z)}$ are plotted in figure 9. (The function $\Phi(z)$ is found to be more suitable then $F(z)$ for the actual calculation of the stability limits.)

The inviscid solutions φ_1 and φ_2 were obtained as power series in α^2 as follows (section 8 of reference 8):

$$\varphi_1\left(y;\ \alpha^2,\ c,\ M_o^2\right) = (w - c)\sum_{n=0}^{\infty} \alpha^{2n} h_{2n}\left(y;\ c,\ M_o^2\right) \tag{2.7}$$

$$\varphi_2\left(y;\ \alpha^2,\ c,\ M_o^2\right) = (w - c)\sum_{n=0}^{\infty} \alpha^{2n} k_{2n+1}\left(y;\ c,\ M_o^2\right) \tag{2.8}$$

where for $n \geqq 1$

$$h_{2n}\left(y;\ c,\ M_o^2\right) = \int_{y_1}^{y} \left[\frac{T}{(w - c)^2} - M_o^2\right] dy \int_{y_1}^{y} \frac{(w - c)^2}{T} h_{2n-2}\left(y;\ c,\ M_o^2\right) dy \tag{2.9}$$

and

$$h_o = 1.0$$

and for $n \geqq 1$

$$k_{2n+1}\left(y;\ c, M_o^2\right) = \int_{y_1}^{y} \left[\frac{T}{(w - c)^2} - M_o^2\right] dy \int_{y_1}^{y} \frac{(w - c)^2}{T} k_{2n-1}\left(y;\ c,\ M_o^2\right) dy \tag{2.10}$$

and

$$k_1\left(y;\ c,\ M_o^2\right) = \int_{y_1}^{y} \left[\frac{T}{(w - c)^2} - M_o^2\right] dy$$

The lower limit in the integrals is taken at the surface merely for convenience. When $y > y_c$, the path of integration must be taken below the point $y = y_c$ in the complex y-plane. The power series in α^2 are then uniformly convergent for any finite value of α.

At the surface, the inviscid solutions are readily evaluated

$$\left.\begin{aligned}
\varphi_{11} &= -c \\
\varphi_{11}' &= w_1' \\
\varphi_{21} &= 0 \\
\varphi_{21}' &= -\frac{1}{c}\left(T_1 - M_o^2 c^2\right)
\end{aligned}\right\} \tag{2.11}$$

At the "edge" of the boundary layer, the inviscid solutions are most conveniently expressed as follows:

$$\left.\begin{aligned}
\varphi_{12} &= (1 - c) \sum_{n=0}^{\infty} \alpha^{2n} H_{2n}\left(c, M_o^2\right) \\
\varphi_{22} &= (1 - c) \sum_{n=0}^{\infty} \alpha^{2n} K_{2n+1}\left(c, M_o^2\right) \\
\varphi_{12}' &= (1 - c) \left[\frac{1 - M_o^2(1 - c)^2}{(1 - c)^2}\right] \sum_{n=1}^{\infty} \alpha^{2n} H_{2n-1}\left(c, M_o^2\right) \\
\varphi_{22}' &= (1 - c) \left[\frac{1 - M_o^2(1 - c)^2}{(1 - c)^2}\right] \sum_{n=0}^{\infty} \alpha^{2n} K_{2n}\left(c, M_o^2\right)
\end{aligned}\right\} \tag{2.12}$$

where

$$\left.\begin{aligned}
H_{2n}\left(c,\ M_o^{\,2}\right) &= h_{2n}\left(y_2;\ c,\ M_o^{\,2}\right) \\
H_o &= 1.0 \\
K_{2n+1}\left(c,\ M_o^{\,2}\right) &= k_{2n+1}\left(y_2;\ c,\ M_o^{\,2}\right) \\
H_{2n-1}\left(c,\ M_o^{\,2}\right) &= \left[\frac{1 - M_o^{\,2}(1 - c)^2}{(1 - c)^2}\right]^{-1} h_{2n}'\left(y_2;\ c,\ M_o^{\,2}\right) \\
K_{2n}\left(c,\ M_o^{\,2}\right) &= \left[\frac{1 - M_o^{\,2}(1 - c)^2}{(1 - c)^2}\right]^{-1} k_{2n+1}'\left(y_2;\ c,\ M_o^{\,2}\right) \\
K_o &= 1.0
\end{aligned}\right\} \qquad (2.13)$$

With the aid of equations (2.11), the expression for $E\left(\alpha,\ c,\ M_o^{\,2}\right)$ can be rewritten as follows:

$$E\left(\alpha,\ c,\ M_o^{\,2}\right) = \frac{1}{1 + \lambda(c)} \frac{w_1'\left(\varphi_{22}' + \beta\varphi_{22}\right)}{w_1'\left(\varphi_{22}' + \beta\varphi_{22}\right) + \frac{T_1}{c}\left(\varphi_{12}' + \beta\varphi_{12}\right)} \qquad (2.14)$$

where

$$\lambda(c) = \frac{w_1'\left(y_c - y_1\right)}{c} - 1 \qquad (2.15)$$

The relation (2.2) between the phase velocity and the wave length is brought into a form more suitable for the calculation of the stability limits by making use of the fact that for real values of c the imaginary part of $E\left(\alpha,\ c,\ M_o^{\,2}\right)$ is contributed largely by the integral $K_1\left(c, M_o^2\right)$. (The procedure to be followed is identical with that used by Lin in the limiting case of the incompressible fluid (reference 5, part III).) Define the function $\Phi(z)$ by the relation

$$\Phi(z) = \frac{1}{1 - F(z)} \tag{2.16}$$

Then,

$$\Phi(z) = \frac{1}{1 - E} = (1 + \lambda) \frac{(u + iv)}{1 + \lambda(u + iv)} \tag{2.17}$$

where

$$u + iv = 1 + \frac{w_1{}'c}{T_1}\left(\frac{\varphi_{22}{}' + \beta\varphi_{22}}{\varphi_{12}{}' + \beta\varphi_{12}}\right) \tag{2.18}$$

Equation (2.17) is equivalent to the two real relations

$$\Phi_i(z) = \frac{(1 + \lambda)v}{(1 + \lambda u)^2 + \lambda^2 v^2} \tag{2.19}$$

$$\Phi_r(z) = (1 + \lambda)\left[\frac{u(1 + \lambda u) + \lambda v^2}{(1 + \lambda u)^2 + \lambda^2 v^2}\right] \tag{2.20}$$

The real and imaginary parts of $\Phi(z)$ are plotted against z in figure 9.

The dominant term in the imaginary part of the right-hand side of equation (2.18), which involves $K_1\left(c, M_o^2\right)$, is extracted by means of straightforward algebraic transformations. Relation (2.18) becomes

$$u + iv = \frac{w_1'c}{T_1}\left[\left(K_1 + \frac{T_1}{w_1'c}\right) + \left(\frac{1 - \alpha^2 H_2}{\alpha}\right)\frac{\frac{\sqrt{1 - M_0^2(1 - c)^2}}{(1 - c)^2}\left(1 - \sum_{n=1}^{\infty}\alpha^{2n}N_{2n}\right) - \sum_{n=1}^{\infty}\alpha^{2n+1}N_{2n+1}}{\left(1 - \sum_{n=2}^{\infty}\alpha^{2n}M_{2n}\right) + \frac{\sqrt{1 - M_0^2(1 - c)^2}}{(1 - c)^2}\left(\alpha H_1 - \sum_{n=1}^{\infty}\alpha^{2n+1}M_{2n+1}\right)}\right] \tag{2.21}$$

where

$$N_2 \equiv H_2$$

and for $n \geqq 3$

$$N_n = K_1 H_{n-1} - K_n \tag{2.22a}$$

and

$$M_n = E_2 H_{n-2} - E_n \tag{2.22b}$$

When c is real,

$$v \approx \frac{w_1'c}{T_1}\,\text{I.P. } K_1$$

for those values of α and c that occur in the stability calculations. (This approximation is justified later in appendix A.) The imaginary part of the integral $K_1\left(c, M_0^2\right)$ is readily computed. It is found that

$$\text{I.P. } K_1\left(c, M_o^2\right) \approx -\pi \frac{T_c^2}{\left(w_c'\right)^3}\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c}$$

$$= -\pi \frac{T_c}{\left(w_c'\right)^2}\left(\frac{w_c''}{w_c'} - \frac{T_c'}{T_c}\right) \tag{2.23}$$

Now $\lambda(c)$ is generally quite small, therefore $\Phi_i(z)$ can be taken equal to $v(c)$ and $\Phi_r(z)$ can be taken equal to u as a zeroth approximation. From equations (2.19) and (2.20), when c is real

$$\Phi_i^{(0)}\left(z^{(0)}\right) = v = -\frac{\pi w_1' c}{T_1} \frac{T_c}{\left(w_c'\right)^2}\left(\frac{w_c''}{w_c'} - \frac{T_c'}{T_c}\right) \tag{2.24}$$

$$u^{(0)} = \Phi_r^{(0)}\left(z^{(0)}\right) \tag{2.25}$$

By equation (2.24), $z^{(0)}$ is related to c with the aid of figure 9; and by equation (2.25), $u^{(0)}$ is also related to c. The quantity αR is connected with c by means of the identity

$$\alpha R = \frac{\nu_c}{w_c'(1 + \lambda)^3}\left(\frac{z w_1'}{c}\right)^3 \tag{2.26}$$

and the corresponding values of α are obtained from equation (2.21) (slightly transformed) by a method of successive approximations.

Thus,

$$\alpha = \frac{\frac{w_1'c}{T_1}\left(1 - \alpha^2 H_2\right)\left[\frac{\sqrt{1 - M_o^2(1 - c)^2}}{(1 - c)^2}\left(1 - \sum_{n=1}^{\infty} \alpha^{2n} N_{2n}\right) - \sum_{n=1}^{\infty} \alpha^{2n+1} N_{2n+1}\right]}{(u - L)\left[1 - \sum_{n=2}^{\infty} \alpha^{2n} M_{2n} + \frac{\sqrt{1 - M_o^2(1 - c)^2}}{(1 - c)^2}\left(\alpha H_1 - \sum_{n=1}^{\infty} \alpha^{2n+1} M_{2n+1}\right)\right]} \tag{2.27}$$

where

$$L = \frac{w_1'c}{T_1} \text{R.P.}\left(K_1 + \frac{T_1}{w_1'c}\right)$$

(The symbols M_k and N_k now designate the real parts of the integrals M_k and N_k.) The iteration process is begun by taking a suitable initial value of α on the right-hand side of equation (2.27). The methods adopted for computing these integrals when the mean velocity-temperature profile is known are described in appendixes A to C.

For greater accuracy, the values of z and u for a given real value of c are computed by successive approximations. From equations (2.19) and (2.20),

$$\phi_i^{(n+1)}\left(z^{(n+1)}\right) = \frac{(1 + \lambda)v}{\left(1 + \lambda u^{(n)}\right)^2 + \lambda^2 v^2} \tag{2.28}$$

$$u^{(n+1)} = \phi_r^{(n+1)}\left(z^{(n+1)}\right)\left[\frac{\left(1 + \lambda u^{(n)}\right)^2 + \lambda^2 v^2}{(1 + \lambda)\left(1 + \lambda u^{(n)}\right)}\right] - \frac{\lambda v^2}{1 + \lambda u^{(n)}} \tag{2.29}$$

The value of v is always approximated by relation (2.24).

Curves of wave number against Reynolds number for the neutral disturbance have been calculated for 10 representative cases (fig. 4), that is, insulated surface at Mach numbers of 0, 0.50, 0.70, 0.90, 1.10, and 1.30 and heat transfer across the solid surface at a Mach number of 0.70 with values of the ratio of surface temperature to free-stream temperature T_1 of 0.70, 0.80, 0.90, and 1.25. (It is found more desirable to base the nondimensional wave number and the Reynolds number on the momentum thickness θ, which is a direct measure of the skin friction, rather than on the boundary-layer thickness δ, which is somewhat indefinite.)

In figure 5 the minimum critical Reynolds number $R_{\theta_{cr_{min}}}$, or the stability limit, is plotted against Mach number for the insulated surface; and in figure 6(a) $R_{\theta_{cr_{min}}}$ is plotted against T_1 for the cooled or heated surface at a Mach number of 0.70. The marked stabilizing influence of a withdrawal of heat from the fluid is clearly evident. Discussion of the physical significance of these numerical results is reserved until after general criterions for the stability of the laminar boundary layer have been obtained.

3. DESTABILIZING INFLUENCE OF VISCOSITY AT VERY LARGE REYNOLDS NUMBERS; EXTENSION OF HEISENBERG'S CRITERION TO THE COMPRESSIBLE FLUID

The numerical calculation of the limits of stability for several particular cases gives some indication of the effects of free-stream Mach number and thermal conditions at the solid surface on the stability of the laminar boundary layer. It would be very desirable, however, to establish general criterions for laminar instability. For the incompressible fluid, Heisenberg has shown that the influence of viscosity is generally destabilizing at very large Reynolds numbers (reference 1). His criterion can be stated as follows: If a neutral disturbance of nonvanishing phase velocity and finite wave length exists in an inviscid fluid ($R \rightarrow \infty$) for a given mean velocity distribution, a disturbance of the same wave length is unstable, or self-excited, in the real fluid at very large (but finite) Reynolds numbers.

The same conclusion can be drawn from Prandtl's discussion of the energy balance for small disturbances in the laminar boundary layer (reference 12).

Heisenberg's criterion is established for subsonic disturbances in the laminar boundary layer of a compressible fluid by an argument quite similar to that which he gave originally for the incompressible fluid and which was later supplemented by Lin (reference 5, part III).

At very large Reynolds numbers, the relation (2.1) between the phase velocity and the wave length can be considerably simplified. When λ is finite and c does not vanish, $|z| >> 1$ at large Reynolds numbers. The asymptotic behavior of the Tietjens function $F(z)$ as $|z| \longrightarrow \infty$ is given by (reference 5, part I)

$$\left(y_1 - y_c\right) F(z) = \frac{-e^{\pi i/4}}{\sqrt{\alpha \dfrac{R}{\upsilon_c} c}} \tag{3.1}$$

and the relation (2.1) becomes

$$\left(y_1 - y_c\right) E\left(\alpha, c, M_o^2\right) = E_1\left(\alpha, c, M_o^2\right) = \frac{-e^{\pi i/4}}{\sqrt{\alpha \dfrac{R}{\upsilon_c} c}} \tag{3.2}$$

where $E\left(\alpha, c, M_o^2\right)$ is given by equation (2.14).

Suppose that a neutral disturbance of nonvanishing wave number $\alpha_s = \dfrac{2\pi}{\lambda_s}$ and phase velocity $c_s > 1 - \dfrac{1}{M_o}$ exists in the inviscid fluid (limiting case of an infinite Reynolds number). The phase velocity c is a continuous function of R, and for a disturbance of given wave number α_s the value of c at very large Reynolds numbers will differ from c_s by a small increment Δc. Both sides of equation (3.2) can be developed in a Taylor's series in Δc, and an expression for Δc can be obtained as follows:

$$E_1\left(\alpha, c, M_o^2\right) = E_1\left(\alpha_s, c_s, M_o^2\right) + \left(\frac{\partial E_1}{\partial c}\right)_{c_s, \alpha_s} \Delta c + \ldots$$

$$= \frac{-e^{\pi i/4}}{\sqrt{\alpha_s \dfrac{R}{\upsilon_{c_s}} c_s}} \left[1 + O(\Delta c)\right] \tag{3.3}$$

The boundary condition

$$\varphi_{22}'\left(\alpha_s, c_s, M_o^2\right) + \beta_s\varphi_{22}\left(\alpha_s, c_s, M_o^2\right) = 0 \tag{3.4}$$

must be satisfied for the inviscid neutral disturbance, and the function $E_1\left(\alpha_s, c_s, M_o^2\right)$ vanishes (equation 2.14). Recognizing that

$$\left(\frac{\partial E_1}{\partial c}\right)_{c_s,\alpha_s} \gg \frac{1}{\sqrt{\alpha_s \frac{R}{\nu_{c_s}} c_s}}$$

reduces equation (3.3) for Δc to the form

$$\Delta c = \frac{-e^{\pi i/4}}{\sqrt{\alpha_s \frac{R}{\nu_{c_s}} c_s \left(\frac{\partial E_1}{\partial c}\right)_{c_s,\alpha_s}}} \tag{3.5}$$

From equation (2.14),

$$\left(\frac{\partial E_1}{\partial c}\right)_{c_s,\alpha_s} = -\frac{c_s^2}{T_1} \frac{\left\{\frac{\partial}{\partial c}\left[\varphi_{22}'\left(\alpha_s, c, M_o^2\right) + \beta\varphi_{22}\left(\alpha_s, c, M_o^2\right)\right]\right\}_{c=c_s}}{\varphi_{12}'\left(\alpha_s, c_s, M_o^2\right) + \beta_s\varphi_{12}\left(\alpha_s, c_s, M_o^2\right)} \tag{3.6}$$

By equations (2.12) and the boundary condition (3.4), the quantity $\left(\frac{\partial E_1}{\partial c}\right)_{c_s,\alpha_s}$ is evaluated as follows:

$$\left(\frac{\partial E_1}{\partial c}\right)_{c_s,\alpha_s} = -\frac{c_s^2}{T_1}\,\frac{(1-c_s)^2\sum_{n=1}^{\infty}\alpha_s^{2n-1}K_{2n-1}'\left(c_s,M_o^2\right)+\sqrt{1-M_o^2(1-c_s)^2}\sum_{n=1}^{\infty}\alpha_s^{2n}K_{2n}'\left(c_s,M_o^2\right)+\frac{2-M_o^2(1-c_s)^2}{\sqrt{1-M_o^2(1-c_s)^2}}\,\frac{1}{(1-c_s)}\sum_{n=0}^{\infty}\alpha_s^{2n}K_{2n}\left(c_s,M_o^2\right)}{(1-c_s)^2\sum_{n=0}^{\infty}\alpha_s^{2n+1}H_{2n}\left(c_s,M_o^2\right)+\sqrt{1-M_o^2(1-c_s)^2}\sum_{n=1}^{\infty}\alpha_s^{2n}H_{2n-1}\left(c_s,M_o^2\right)} \tag{3.7}$$

where the primes now denote differentiation with respect to c. For small values of c_s and α_s, the quantity $\left(\frac{\partial E_1}{\partial c}\right)_{c_s,\alpha_s}$ is given approximately by the relation

$$\left(\frac{\partial E_1}{\partial c}\right)_{c_s,\alpha_s} = -\frac{c_s^2}{T_1}\left[\frac{2-M_o^2(1-c_s)^2}{\alpha_s(1-c_s)^3\sqrt{1-M_o^2(1-c_s)^2}}+K_1'\left(c_s,\ M_o^2\right)\right] \tag{3.8}$$

and the expression for Δc is

$$\Delta c = \frac{T_1}{\sqrt{\alpha_s \frac{R}{\upsilon_{c_s}} c_s^5}} \frac{\alpha_s \sqrt{1 - M_o^2(1 - c_s)^2}\ e^{\pi i/4}}{\frac{2 - M_o^2(1 - c_s)^2}{(1 - c_s)^3} + \alpha_s \sqrt{1 - M_o^2(1 - c_s)^2}\ K_1'(c_s, M_o^2)} \tag{3.9}$$

Evaluation of the integral $K_1(c, M_o^2)$ yields the following result:

$$K_1(c, M_o^2) = -\frac{T_1}{w_1'c} + \frac{T_c^2}{(w_c')^3}\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c}(\ln c - i\pi) + O(1) \tag{3.10}$$

Since the quantity $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c_s}$ vanishes (reference 8), differentiation of equation (3.10) gives

$$K_1'(c_s, M_o^2) = \frac{T_1}{w_1'c_s^2} + \left(\frac{\partial}{\partial c}\left\{\frac{T_c^2}{(w_c')^3}\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c}\right\}\right)_{c=c_s}(\ln c_s - i\pi) + O(1) \tag{3.11}$$

Thus, $K_1'(c_s, M_o^2)$ is approximately real and positive for small values of c_s. With $c_s > 1 - \frac{1}{M_o}$, <u>I.P. Δc must also be positive</u> (equation (3.9)); therefore, a subsonic disturbance of wave length $\lambda_s \neq 0$, which is neutral in the inviscid compressible fluid, is self-excited in the real compressible fluid at very large (but finite) Reynolds numbers.

In reference 8, it was proved that a neutral subsonic boundary-layer disturbance of nonvanishing phase velocity and finite wave length exists in an inviscid compressible fluid only if the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ vanishes for some value of $w > 1 - \frac{1}{M_o}$. If this condition is satisfied, then self-excited subsonic disturbances also exist in the fluid, and the laminar boundary layer is unstable in the limiting case of an infinite Reynolds number. By the extension of Heisenberg's criterion to the compressible fluid, it can be seen that, far from stabilizing the flow, the small viscosity in the real fluid has, on the contrary, a destabilizing influence at very large Reynolds numbers. Thus, any laminar boundary-layer flow in a viscous conductive gas for which the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ vanishes for some value of $w > 1 - \frac{1}{M_o}$ is unstable at sufficiently high (but finite) Reynolds numbers.

Unless the condition $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right) = 0$ for some value of $w > 1 - \frac{1}{M_o}$ is satisfied, all subsonic disturbances of finite wave length are damped in the limiting case of infinite Reynolds number, and the inviscid flow is stable. Since the effect of viscosity is destabilizing at very large Reynolds numbers, however, a laminar boundary flow that is stable in the limit of infinite Reynolds number is not necessarily stable at large Reynolds numbers when the viscosity of the fluid is considered. (See fig. 4(l).) In fact, for the incompressible fluid, Lin has shown that every laminar boundary-layer flow is unstable at sufficiently high Reynolds numbers, whether or not the vorticity gradient $\frac{d^2w}{dy^2}$ vanishes (reference 5, part III). In order to settle this question for the compressible fluid in general terms, the relation (2.1) between the complex phase velocity and the wave length at large Reynolds numbers must now be studied for flows in which the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ does not vanish for any value of $w > 1 - \frac{1}{M_o}$.

4. STABILITY OF LAMINAR BOUNDARY LAYER AT LARGE REYNOLDS NUMBERS

The neutral subsonic disturbance marks a possible "boundary" between the damped and the self-excited disturbance, that is, between stable and unstable flow. Thus, the general conditions under which self-excited disturbances exist in the laminar boundary layer at large Reynolds numbers can be determined from a study of the behavior of the curve of α against R for the neutral disturbances. When the mean free-stream velocity is subsonic $\left(M_0 < 1\right)$, the physical situation for the subsonic disturbances at large Reynolds numbers is quite similar to the analogous situation for the incompressible fluid. The curve of α against R for the neutral disturbances can be expected to have two distinct asymptotic branches that enclose a region of instability in the α,R-plane, regardless of the local distribution of mean velocity and mean temperature across the boundary layer. When the mean free-stream velocity is supersonic $\left(M_0 > 1\right)$ the situation is somewhat different; under certain conditions (soon to be defined) a neutral or a self-excited subsonic disturbance $\left(c > 1 - \frac{1}{M_0}\right)$ cannot exist at any value of the Reynolds number. For this reason, it is more convenient to study the case of subsonic and supersonic free-stream velocity separately.

a. Subsonic Free-Stream Velocity $\left(M_0 < 1\right)$

The asymptotic behavior at large Reynolds numbers of the curve of α against R for the neutral disturbances is determined by the relations (2.19) to (2.22) between α, R, and c for real values of c. For small values of α and c, these relations are given approximately by

$$v(c) = \Phi_i(z) = -\frac{\pi w_1'c}{T_1}\frac{T_c^2}{\left(w_c'\right)^3}\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c} \tag{4.1}$$

$$u = \Phi_r(z) \tag{4.2}$$

$$R = \frac{T_1^{1.76}}{\alpha}\left(\frac{z}{c}\right)^3 \left(w_1'\right)^2 \tag{4.3}$$

$$\alpha = \frac{w_1'c}{T_1}\frac{1}{u}\sqrt{1 - M_o^2(1 - c)^2} \tag{4.4}$$

As $R \longrightarrow \infty$, either $z \longrightarrow \infty$ or z remains finite while both α and c approach 0. These two possibilities correspond to two asymptotic branches of the curve of α against R.

Lower branch.- If z remains finite as $R \longrightarrow \infty$, then $c \longrightarrow 0$; and by equation (4.1), $\Phi_i(z) \longrightarrow 0$. Therefore, $z \longrightarrow 2.29$ while $u \longrightarrow 2.29$ (fig. 9). From equations (4.3) and (4.4), along the lower branch of the curve of α against R for neutral stability

$$R \approx \frac{\left(w_1'\right)^5\left(1 - M_o^2\right)^{3/2}}{T_1^{1.24}}\frac{1}{\alpha^4} \tag{4.5}$$

$$c \approx 2.29\frac{T_1}{w_1'\sqrt{1 - M_o^2}}\alpha \tag{4.6}$$

and $\alpha \longrightarrow 0$ at large Reynolds numbers (fig. 4(l)).

Upper branch.- Along the upper branch of the curve of α against R for neutral stability, $z \longrightarrow \infty$ and

$$\Phi_i(z) = -\frac{\pi w_1'c}{T_1}\frac{T_c^2}{\left(w_c'\right)^3}\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c} \longrightarrow \frac{1}{\sqrt{2z^3}} \approx \frac{w_1'}{\sqrt{2\alpha\frac{R}{\upsilon_c}c^3}} \tag{4.7}$$

while $u \longrightarrow 1.0$ (fig. 9 and equation (4.2)). If the quantity $\frac{d}{dy}\left(\frac{w'}{T}\right)$ does not vanish for any value of $w > 0$, then by equation (4.7) c must approach zero as $z \longrightarrow \infty$. Along this branch,

$$R \approx \frac{\left(w_1'\right)^{11}}{2\pi^2 T_1^{5.24}} \frac{\left(1 - M_o^2\right)^{5/2}}{\left\{\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_1\right\}^2} \frac{1}{\alpha^6} \tag{4.8}$$

$$c \approx \frac{T_1}{w_1'\sqrt{1 - M_o^2}}\,\alpha \tag{4.9}$$

and $\alpha \longrightarrow 0$ at large Reynolds numbers (fig. 4(l)).

On the other hand, if $\frac{d}{dy}\left(\frac{w'}{T}\right)$ vanishes for some value of $w \doteq c_s > 0$, then by equation (4.7), $c \longrightarrow c_s$ and $\alpha \longrightarrow \alpha_s$ as both z and R approach ∞. Now,

$$\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c} = \left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1 \frac{c - c_s}{w_1'} + \left\{\left[\frac{d^3}{dy^3}\left(\frac{w'}{T}\right)\right]_1 - \left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right] \frac{w_1''}{w_1'}\right\} \frac{c^2 - c_s^2}{2\left(w_1'\right)^2} + \ldots \tag{4.10}$$

If $\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1$ does not vanish (see appendix D), then by equations (4.4) and (4.7), along the upper branch of the curve of α against R for the neutral disturbances,

$$R \approx \frac{\left(w_1'\right)^8}{2\pi^2 T_1^{0.24}} \frac{1}{\left\{\left[\frac{d^2}{dy^2}\left(\frac{w'}{T_1}\right)\right]_1\right\}^2} \frac{1}{\alpha c^5} \frac{1}{\left(c - c_s\right)^2} \tag{4.11}$$

$$\alpha \approx \frac{w_1' c}{T_1}\sqrt{1 - M_o^2(1 - c)^2} \tag{4.12}$$

and $c \longrightarrow c_s \neq 0$, $\alpha \longrightarrow \alpha_s \neq 0$ at large Reynolds numbers (figs. 4(k) and 4(l)). If $\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1$ vanishes, the relation (4.11) is replaced by

$$R \approx \frac{2\left(w_1'\right)^{10}}{\pi^2 T_1^{0.24}} \frac{1}{\left\{\left[\frac{d^3}{dy^3}\left(\frac{w'}{T}\right)\right]_1\right\}^2} \frac{1}{\alpha c^5} \frac{1}{\left(c^2 - c_s^2\right)^2} \tag{4.13}$$

which reduces to the relation obtained by Lin in the limiting case of an incompressible fluid when $M_o \longrightarrow 0$, the solid boundary is insulated, and $w'' = 0$ for some value of $w = c_s > 0$. (See equation (12.22) of reference 5, part III.)

If the quantity $\frac{d}{dy}\left(\frac{w'}{T}\right)$ vanishes <u>at</u> the solid boundary (that is, for $w = 0$), it can be shown from the equations of motion (appendix D) that $\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1$ is always positive - except in the limiting case of an incompressible fluid. For small values of y, the quantities $\frac{d}{dy}\left(\frac{w'}{T}\right)$ and $\frac{w'}{T}$ are both positive and increasing.

For large values of y, however, $\frac{w'}{T} \longrightarrow 0$, physically; therefore $\frac{w'}{T}$ must have a maximum, or $\frac{d}{dy}\left(\frac{w'}{T}\right) = 0$ for some value of $w > 0$, and this case is no different from the general case treated in the preceding paragraph. In the limiting case of an incompressible fluid, when w'' vanishes at the surface, $w_c'' = w_1^{iv} \frac{c^2}{2(w_1')^2}$ since w_1''' always vanishes in this case. From equation (4.8) the relation between α and R along the upper branch of the neutral stability curve is therefore

$$R \approx \frac{2(w_1')^{19}}{\pi^2} \frac{1}{(w_1^{iv})^2} \frac{1}{\alpha^{10}} \tag{4.14}$$

which is identical with equation (12.19) in reference 5, part III.

Thus, regardless of the behavior of the quantity $\frac{d}{dy}\left(\frac{w'}{T}\right)$ - regardless of the local distribution of mean velocity and mean temperature across the boundary layer - when $M_o < 1$, the curve of α against R for the neutral disturbances has two distinct branches at large Reynolds numbers. From physical considerations, all subsonic disturbances must be damped when the wave length is sufficiently small (α large) or the Reynolds number is sufficiently low. Consequently, the two branches of the curve of α against R for the neutral disturbances must join eventually, and the region between them in the α,R-plane is a region of instability; that is, at a given value of the Reynolds number, subsonic disturbances with wave lengths lying between two critical values λ_1 and λ_2 (α_1 and α_2) are self-excited. Thus, when $M_o < 1$, any laminar boundary-layer flow in a viscous conductive gas is unstable at sufficiently high (but finite) Reynolds numbers.

The lower branch of the curve of α against R for the neutral disturbances is virtually unaffected by the distribution of $\frac{d}{dy}\left(\frac{w'}{T}\right)$ across the boundary layer, but for the upper branch the behavior of the quantity $\frac{d}{dy}\left(\frac{w'}{T}\right)$ is decisive. When $\frac{d}{dy}\left(\frac{w'}{T}\right) = 0$ for some value of $w = c_s > 0$, the neutral subsonic disturbance passes continuously into the characteristic inviscid disturbance $c = c_s$ and $\alpha = \alpha_s$ as $R \longrightarrow \infty$. This result is in accordance with the results obtained in reference 9 for the inviscid compressible fluid and is in agreement with Heisenberg's criterion. In addition, all subsonic disturbances of finite wave length $\lambda > \lambda_s = \frac{2\pi}{\alpha_s}$ (and nonvanishing phase velocity $0 < c_r < c_s$) are self-excited in the limiting case of infinite Reynolds number. On the other hand, when $\frac{d}{dy}\left(\frac{w'}{T}\right)$ does not vanish for any value of $w > 0$, then except for the "singular" neutral disturbance of zero phase velocity and infinite wave length ($c = 0$ and $\alpha = 0$), all disturbances are damped in the inviscid compressible fluid. This singular neutral disturbance can be regarded as the limiting case of the neutral subsonic disturbance in a real compressible fluid as $R \longrightarrow \infty$.

b. Supersonic Free-Stream Velocity $\left(M_0 > 1\right)$

When the velocity of the free stream is supersonic, the subsonic boundary-layer disturbances must satisfy not only the differential equations and the boundary conditions of the problem but also the physical requirement that $c_r > 1 - \frac{1}{M_0}$. The asymptotic behavior at large Reynolds numbers of the curve of α against R for the neutral subsonic disturbances is determined by the approximate relations (4.1) to (4.4), with the additional restriction that $c > 1 - \frac{1}{M_0}$. As $c \longrightarrow 1 - \frac{1}{M_0}$, $\alpha \longrightarrow 0$ by equation (4.4); therefore $R \longrightarrow \infty$ by equation (4.3). The corresponding value (or values) of z is determined by equation (4.1) as follows:

$$\Phi_1(z) = v(c) = v\left(1 - \frac{1}{M_0}\right) = \frac{-\pi w_1'\left(1 - \frac{1}{M_0}\right)}{T_1} \left[\frac{T^2}{(w')^3} \frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c=1-\frac{1}{M_0}} \qquad (4.15)$$

Now from physical considerations, $\frac{d}{dy}\left(\frac{w'}{T}\right) < 0$ for large values of y. Therefore, if $\frac{d}{dy}\left(\frac{w'}{T}\right) = 0$ (changes sign) for some value of $w = c_s > 1 - \frac{1}{M_o}$, then, in general, $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=1-\frac{1}{M_o}} > 0$ and $\Phi_i(z)_{c=1-\frac{1}{M_o}} < 0$ (equation (4.15)). From figure 9, it can be seen that in this case there is only one value of z (z_1, say) corresponding to the value of $\Phi_i(z)$ given by equation (4.15). From equations (4.2) to (4.4), along the lower branch of the curve of α against R for the neutral disturbances,

$$R \approx \frac{T_1^{1.76}(w_1')^2 z_1^3}{\left(1 - \frac{1}{M_o}\right)^3} \frac{1}{\alpha} \tag{4.16}$$

$$\alpha \approx \frac{w_1'\left(1 - \frac{1}{M_o}\right)\sqrt{2M_o}}{T_1 u_1} \sqrt{c - \left(1 - \frac{1}{M_o}\right)} \tag{4.17}$$

and $c \longrightarrow 1 - \frac{1}{M_o}$ at large Reynolds numbers (fig. 4(k)). The upper branch of the curve in this case is given by equations (4.11) and (4.12), or by equations (4.13) and (4.12) if $\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1$ vanishes, with $c \longrightarrow c_s > 1 - \frac{1}{M_o}$ and $\alpha \longrightarrow \alpha_s \neq 0$.

If $\frac{d}{dy}\left(\frac{w'}{T}\right)$ vanishes for $w = 1 - \frac{1}{M_o}$, then $z \longrightarrow \infty$ as $R \longrightarrow \infty$ along the upper branch of the curve of α against R for the neutral disturbances, and $\Phi_1(z) \longrightarrow \frac{w_1'}{\sqrt{2\alpha \frac{R}{\nu_c} c^3}}$. Now $\alpha \longrightarrow 0$ as $c \longrightarrow 1 - \frac{1}{M_o}$ in this case also (equation (4.17) with $u_1 = 1.0$) so that

$$R \approx \frac{2\left(w_1'\right)^{12}}{\pi^2 T_1^{4.24}} \frac{M_o^2}{1 - \frac{1}{M_o}} \frac{1}{\left\{\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1\right\}^2} \frac{1}{\alpha^5} \tag{4.18}$$

Along the lower branch of the curve of α against R at large Reynolds numbers, α, R, and c are connected by equations (4.16) and (4.17), with $z_1 = 2.29$ and $u_1 = 2.29$. In spite of the fact that $\frac{d}{dy}\left(\frac{w'}{T}\right) = 0$ for $w = 1 - \frac{1}{M_o}$, a neutral sonic disturbance $\left(c = 1 - \frac{1}{M_o}\right)$ of finite wave length does not exist in the inviscid fluid unless $K_1(c) = \int_0^\infty \left[\frac{T}{(w - c)^2} - M_o^2\right] dy$ is positive. (See section 10 of reference 8.) Calculation shows that $K_1(c)$ is almost always negative (equation (3.11)); therefore, in general, the sonic disturbance of infinite wave length $(\alpha = 0)$ with constant phase across the boundary layer exists only in the inviscid fluid $(R \longrightarrow \infty)$.

If $\frac{d}{dy}\left(\frac{w'}{T}\right)$ does not vanish for any value of $w \geqq 1 - \frac{1}{M_o}$, it is certain that $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c=1-\frac{1}{M_o}} < 0$ and by equation (4.15)

$\Phi_1(z)_{c=1-\frac{1}{M_o}} > 0$. When $v_{1-\frac{1}{M_o}} < 0.580$ (approx.), there are two values of z $\left(z_2 \text{ and } z_3, \text{ say, with } z_3 > z_2\right)$ corresponding to the value of $\Phi_1(z)$ given by equation (4.15). (See fig. 9.) Along the two asymptotic branches of the curve of α against R for the neutral disturbances, α, R, and c are connected by relations of the form of equations (4.16) and (4.17), with z and u replaced by z_2 and u_2, respectively, along the lower branch and by z_3 and u_3, respectively, along the upper branch. At a given value of the Mach number, the value of $v_{1-\frac{1}{M_o}}$ is controlled by the thermal conditions at the solid surface. (See section 6.) When these conditions are such that $v_{1-\frac{1}{M_o}} \approx 0.580$, then $z_2 = z_3$, and the two asymptotic branches of the curve of α against R for the neutral disturbances coincide. When $v_{1-\frac{1}{M_o}} \geqq 0.580$ (approx.), it is impossible for a neutral or a self-excited subsonic disturbance to exist in the laminar boundary layer of a viscous conductive gas at any value of the Reynolds number. In other words, if $v_{1-\frac{1}{M_o}} \geqq 0.580$ (approx.), the laminar boundary layer is stable at all values of the Reynolds number. (Of course, in any given case, the critical conditions beyond which only damped subsonic disturbances exist can be calculated more accurately from the relations (2.28) and (2.29). See section 5 on minimum critical Reynolds number.)

The preceding conclusion can also be deduced, at least qualitatively, from the results of a study of the energy balance for a neutral subsonic disturbance in the laminar boundary layer. A neutral subsonic disturbance can exist only when the destabilizing effect of viscosity near the solid surface, the damping effect of viscosity in the fluid, and the energy transfer between mean flow and disturbance in the vicinity of the inner "critical layer" all balance out to give a zero (average) net rate of change of the energy of the disturbance. (See Schlichting's discussion for incompressible fluid in reference 4.) In reference 8 it is shown that the sign and magnitude of the phase shift in $u^{*\prime}$ through the inner "critical layer" at $w = c$ is determined by the sign and magnitude of the quantity $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c}$. The corresponding apparent shear stress $\tau_c^* = \overline{-\rho^*}\ \overline{u^{*\prime}v^{*\prime}}$, which is zero for $w < c$ in the inviscid compressible fluid, is given by the following expression for $w > c$ (reference 8).

$$\tau_c^* = \overline{\rho_o^*}\left(\overline{u_o^*}\right)^2 \frac{\alpha}{2}\pi \frac{T_c^2}{\left(w_c'\right)^3}\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c} \tag{4.19}$$

If the quantity $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c}$ is negative, the mean flow absorbs energy from the disturbance; if $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c}$ is positive, energy passes from the mean flow to the disturbance. In the real compressible fluid, the thickness of the inner critical layer in which the viscous forces are important is of the order of $\frac{1}{\left(\alpha \frac{R}{\nu_c}\right)^{1/3}}$, and the phase shift in $u^{*\prime}$ is actually brought about by the effects of viscous diffusion (of the quantity $\rho\frac{dw}{dy}$) through this layer.

As shown by Prandtl (reference 12), the destabilizing effect of viscosity near the solid surface is to shift the phase of the "frictional" component $u_{fr}^{*\prime}$ of the disturbance velocity against the phase of the "frictionless" or "inviscid" component $u_{inv}^{*\prime}$ in a thin layer of fluid of thickness of the order of $\sqrt{\frac{1}{\alpha c \frac{R}{\nu_1}}}$. By continuity, the associated normal component $v_{fr}^{*\prime}$ is of the order of $\left|\frac{\partial u^{*\prime}}{\partial x^*}\right|\sqrt{\frac{1}{\alpha c \frac{R}{\nu_1}}} \approx \left|u_{inv}^{*\prime}\right|_1\sqrt{\frac{\alpha}{c \frac{R}{\nu_1}}}$. (It was shown in part 1 of reference 8, that for large values of αR the "frictional" components of the disturbance also satisfy the continuity relation, $\frac{\partial u^{*'}}{\partial x^*} + \frac{\partial v^{*'}}{\partial y^*} = 0$ in the compressible fluid.) The corresponding apparent shear stress $\tau_1^* = \overline{-\rho_1^*}\ \overline{u^{*'}v^{*'}}$ is given by the expression

$$\tau_1^* \approx \overline{\rho_o^*}\left(\overline{u_o^*}\right)^2 \frac{\overline{\rho_1^*}}{\overline{\rho_o^*}}\left[\left(\frac{u_{inv}^{*'}}{\overline{u_o^*}}\right)_1\right]^2 \sqrt{\frac{\alpha}{c\frac{R}{\nu_1}}} \tag{4.20}$$

But from equations (2.11)

$$\left|\left(\frac{u_{inv}^{*'}}{\overline{u_o^*}}\right)_1\right| \approx \left|\frac{T_1}{T_1 - M_o^2c^2}\varphi_{21}'\right| = \frac{T_1}{c} \tag{4.21}$$

and

$$\tau_1^* \approx \overline{\rho_o^*}\left(\overline{u_o^*}\right)^2 \frac{T_1}{c^2}\sqrt{\frac{\alpha}{c\frac{R}{\nu_1}}} \tag{4.22}$$

Since the shear stress associated with the destabilizing effect of viscosity near the solid surface and the shear stress near the critical layer act roughly throughout the same region of the fluid, the ratio of the rates of energy transferred $\left(\text{approximately } \int_0^h \tau^* \frac{du^*}{dy^*}\,dy^*\right)$ by the two physical processes is

$$\left|\frac{E_c^*}{E_1^*}\right| \sim \left|\frac{T_c^*}{T_1^*}\right| \approx \frac{\pi}{2}\frac{w_1'c}{T_1}\frac{T_c^2}{(w_c')^3}\left|\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c}\right| z^{3/2} \tag{4.23}$$

$$= \frac{1}{2}\left|v(c)\right| z^{3/2}$$

where

$$z^3 \approx \alpha\frac{R}{\nu_c}\frac{c^3}{(w_1')^2}$$

If the quantity $\frac{d}{dy}\left(\frac{w'}{T}\right)$ is <u>negative</u> and sufficiently large when $w = c_1$, say, then the rate at which energy is absorbed by the mean flow near the inner "critical layer" plus the rate at which the energy of the disturbance is dissipated by viscous action more than counterbalances the rate at which energy passes from the mean flow to the disturbance because of the destabilizing effect of viscosity near the solid surface. Consequently, a neutral subsonic disturbance with the phase velocity $c \geqq c_1$ does not exist; in fact, all subsonic disturbances for which $c \geq c_1$ are <u>damped</u>. When $M_o < 1$, there is always a range of values of phase velocity $0 \leqq c \leqq c_o$ for which the ratio $\left|\frac{E_c^*}{E_1^*}\right|$, given by equation (4.22), is small enough for neutral (and self-excited) subsonic disturbances to exist for Reynolds numbers greater than a certain critical value. When $M_o > 1$, however, because of the physical requirement that $c > 1 - \frac{1}{M_o} > 0$, the possibility exists that for certain thermal conditions at the solid surface the quantity $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c}$ is always sufficiently large negatively (and therefore $\left|\frac{E_c^*}{E_1^*}\right|$ is sufficiently large) so that only damped subsonic disturbances exist at all Reynolds numbers. Of course, if $\frac{d}{dy}\left(\frac{w'}{T}\right)$ vanishes for some value of $w \geqq 1 - \frac{1}{M_o}$, it is certain that $v(c) < 0.580$ for some range of values of the phase velocity $1 - \frac{1}{M_o} \leqq c \leqq c_o$. In that case, neutral and self-excited subsonic disturbances always exist for $R > R_{cr_{min}}$ and the flow is always unstable at sufficiently high Reynolds numbers, in accordance with Heisenberg's criterion as extended to the compressible fluid (section 2).

A discussion of the significance of these results is reserved for a later section (section 6) in which the behavior of the quantity $\frac{d}{dy}\left(\frac{w'}{T}\right)$ will be related directly to the thermal conditions at the solid surface and the free-stream Mach number.

5. CRITERION FOR THE MINIMUM CRITICAL REYNOLDS NUMBER

The object of the stability analysis is not only to determine the general conditions under which the laminar boundary layer is unstable at sufficiently high Reynolds numbers but also, if possible, to obtain some simple criterion for the limit of stability of the flow (minimum critical Reynolds number) in terms of the local distribution of mean velocity and mean temperature across the boundary layer. For plane Couette motion (linear velocity profile) and plane Poiseuille motion (parabolic velocity profile) in an incompressible fluid, Synge (reference 13) was able to prove rigorously that a minimum critical Reynolds number actually exists below which the flow is stable. His proof applies also to the laminar boundary layer in an incompressible fluid, with only a slight modification (reference 5, part III). Such a proof is more difficult to give for the laminar boundary layer in a viscous conductive gas; however, the existence, in general, of a minimum critical Reynolds number can be inferred from purely physical considerations. A study of the energy balance for small disturbances in the laminar boundary layer shows that the ratio of the rate of viscous dissipation to the rate of energy transfer near the critical layer is $1/R$ for a disturbance of given wave length while the energy transfer associated with the destabilizing action of viscosity near the solid surface bears the ratio $1/\sqrt{R}$ to the energy transfer near the critical layer. Thus, the effects of viscous dissipation will predominate at sufficiently low Reynolds numbers and all subsonic disturbances will be damped. The two distinct asymptotic branches of the curve of α against R for the neutral disturbances at large Reynolds numbers must join eventually (section 4) and the flow is stable for all Reynolds numbers less than a certain critical value.

An estimate of the value of $R_{cr_{min}}$, which will serve as a stability criterion, is obtained by taking the phase velocity c to have the maximum possible value c_o for a neutral subsonic disturbance, that is, for $c > c_o$ all subsonic disturbances are damped. This condition is very nearly equivalent to the condition that αR be a minimum, which was employed by Lin for the case of the incompressible fluid (p. 285 of reference 5, part III). The condition $c = c_o$ occurs when $\Phi_i(z)$ is a maximum; that is, when $\Phi_i(z) = 0.580$, $z_o = 3.22$ and $\Phi_r(z_o) = 1.48$ (fig. 9). The corresponding value of $c = c_o$ can be calculated from the relations (2.19) to (2.22). Neglecting terms in λ^2 (λ is usually very small) and taking $u = 1.50$ gives

$$\Phi_i(z) \approx \left[1 - 2\lambda(c)\right] v(c) \tag{5.1}$$

where

$$v(c) = -\pi \frac{w_1'c}{T_1}\left[\frac{T^2}{(w')^3}\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c} \tag{5.2}$$

and

$$\lambda(c) = \frac{w_1'(y_c - y_1)}{c} - 1 \tag{5.3}$$

It is only necessary to plot the quantity $(1 - 2\lambda)v$ against c for a given laminar boundary-layer flow and find the value of $c = c_o$ for which $(1 - 2\lambda)v = 0.580$. The corresponding value of αR is determined from the relation

$$\alpha R = \left[T\left(c_o\right)\right]^{1.76}\left(w_1'\right)^2\left(\frac{z_o}{c_o}\right)^3 \tag{5.4}$$

and this value of αR is very close to the minimum value of αR. A rough estimate of the value of α for $c = c_o$ is given by the following relation (equation (2.27)):

$$\alpha \approx w_1' c_o \sqrt{1 - M_o^2\left(1 - c_o\right)^2} \tag{5.5}$$

This estimated value of α is, in general, too small. The following estimate of $R_{cr_{min}}$ is obtained by making an approximate allowance for this discrepancy and by taking round numbers:

$$R_{cr_{min}} \approx \frac{25\left[T\left(c_o\right)\right]^{1.76} w_1'}{c_o{}^4\sqrt{1 - M_o^2\left(1 - c_o\right)^2}} \tag{5.6}$$

or

$$R_{\theta\,cr_{min}} \approx \frac{17\left[T\left(c_o\right)\right]^{1.76}\left(\frac{\partial w}{\partial \eta}\right)_1}{c_o{}^4\sqrt{1 - M_o^2\left(1 - c_o\right)^2}} \tag{5.7}$$

For zero pressure gradient, the slope of the velocity profile at the surface $\left(\frac{\partial w}{\partial \eta}\right)_1$ is given very closely by (appendix B)

$$\left(\frac{\partial w}{\partial \eta}\right)_1 = \frac{\left(\frac{\partial w}{\partial \eta}\right)_{1_B}}{T_1}$$

$$= \frac{0.332}{T_1}$$

Therefore

$$R_{\theta_{cr_{min}}} \approx \frac{6}{T_1} \frac{\left[T\left(c_o\right)\right]^{1.76}}{c_o \sqrt[4]{1 - M_o^{2}\left(1 - c_o\right)^2}} \tag{5.8}$$

The expression (5.8) is useful as a rough criterion for the dependence of $R_{\theta_{cr_{min}}}$ on the local distribution of mean velocity and mean temperature across the boundary layer. It is immediately evident that $R_{\theta_{cr_{min}}} \longrightarrow \infty$ when $c_o \longrightarrow 1 - \frac{1}{M_o}$. When $\left[(1 - 2\lambda)v\right]_{c=1-\frac{1}{M_o}} \geqq 0.580$, <u>the laminar boundary layer is stable at all values of the Reynolds number</u>. (This condition is an improvement on the stability condition $v_{1-\frac{1}{M_o}} \geqq 0.580$ (approx.) stated in section 4.)

In the following tables and in figures 5 and 6(a) the estimated values of $R_{\theta_{cr_{min}}}$ given by equation (5.8) can be compared with the values of $R_{\theta_{cr_{min}}}$ taken from the calculated curves of α_θ against R_θ for the neutral disturbances. For the insulated surface, the values are

M_0	c_0	$T(c_0)$	$R_{\theta_{cr_{min}}}$ (est.)	$R_{\theta_{cr_{min}}}$ (fig. 4)
0	0.4186	1.0000	195	150
.50	.4400	1.0408	170	136
.70	.4600	1.0782	150	126
.90	.4850	1.1254	129	115
1.10	.5139	1.1803	109	104
1.30	.5450	1.2406	92	92

For the noninsulated surface when $M_0 = 0.70$, the values are

T_1	c_0	$T(c_0)$	$R_{\theta_{cr_{min}}}$ (est.)	$R_{\theta_{cr_{min}}}$ (fig. 4)
0.70	0.1872	0.7712	5377	5150
.80	.2619	.8716	1463	1440
.90	.3394	.9562	524	523
1.25	.5194	1.1449	89	63

The expression (5.8) for $R_{\theta_{cr_{min}}}$ gives the correct order of magnitude and the proper variation of the stability limit with Mach number and with surface temperature at a given Mach number.

The form of the criterion for the minimum critical Reynolds number (equation (5.8)) and the results of the detailed stability calculations for several representative cases (figs. 3 and 4) show that the distribution of the product of the density and the vorticity $\rho\frac{dw}{dy}$ across the boundary layer largely determines the limits of stability of laminar boundary-layer flow. The fact that the "proper" Reynolds number that appears in the boundary-layer stability calculations is based on the kinematic viscosity at the inner critical layer (where the viscous forces are important) rather than in the free stream also enters the problem, but it amounts only to a numerical and not a qualitative change when the usual Reynolds number based on free-stream kinematic viscosity is finally computed. Whether the value of $R_{\theta_{cr_{min}}}$ for a given laminar boundary-layer flow is larger or smaller than the value of $R_{\theta_{cr_{min}}}$ for the Blasius flow, for example, is determined entirely by the distribution of $\rho\frac{dw}{dy}$ across the boundary layer. If the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ is negative and large near the solid surface so that the quantity $(1 - 2\lambda)v(c)$ reaches the value 0.580 when the value of $c = c_o$ is less than 0.4186, the flow is relatively more stable than the Blasius flow. If the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ is positive near the solid surface, so that $(1 - 2\lambda)v(c) = 0.580$ when $w(\text{or } c) > 0.4186$, the flow is relatively less stable than the Blasius flow. Thus, the question of the relative influence on $R_{\theta_{cr_{min}}}$ of the kinematic viscosity at the inner critical layer and the distribution of $\rho\frac{dw}{dy}$ across the boundary layer, which remained open in the concluding discussions of reference 8, is now settled.

The physical basis for the predominant influence on $R_{\theta_{cr_{min}}}$ of the distribution of $\rho\frac{dw}{dy}$ across the boundary layer is to be found in a study of the energy balance for a subsonic boundary-layer disturbance (section 4). The distribution of $\rho\frac{dw}{dy}$ determines the maximum possible value of the phase velocity c_o or the maximum possible distance of the inner critical layer from the solid surface for a neutral subsonic disturbance. The greater the distance of the inner critical layer from the solid surface, the greater (relatively) the rate of energy absorbed by the mean flow from the disturbance in the vicinity of the critical layer (equations (4.21) and (4.22)). When c_o is large, therefore, the energy balance for a neutral subsonic disturbance is achieved only when the destabilizing action of viscosity near the solid surface is relatively large or, in other words, when $\frac{1}{\sqrt{\alpha_o\frac{R_o}{\nu_{c_o}}}} \approx c_o^{3/2}$ is large and the Reynolds number R_o, which is very nearly equal to $R_{cr_{min}}$, is correspondingly small. On the other hand, when c_o is small and the inner critical layer is close to the solid surface, the rate at which energy is absorbed from the disturbance near the critical layer is relatively small and the rate at which energy passes to the disturbance near the solid surface, which is of the order of $\frac{1}{\sqrt{\alpha \frac{R}{\upsilon_c}}}$, is also relatively small for energy balance; consequently $R_{cr_{min}}$ is large.

6. PHYSICAL SIGNIFICANCE OF RESULTS OF STABILITY ANALYSIS

a. General

From the results obtained in the present paper and in reference 8, it is clear that the stability of the laminar boundary layer in a compressible fluid is governed by the action of <u>both</u> viscous and inertia forces. Just as in the case of an incompressible fluid, the stability problem cannot be understood unless the viscosity of the fluid is taken into account. Thus, whether or not a laminar boundary-layer flow is unstable in the inviscid compressible fluid $(R \longrightarrow \infty)$, that is, whether or not the product of the density and the vorticity $\rho\frac{dw}{dy}$ has an extremum for some value of $w > 1 - \frac{1}{M_o}$, there is always some value of the Reynolds number $R_{cr_{min}}$ below which the effect of viscous dissipation predominates and the flow is stable. On the other hand, at very large Reynolds numbers the influence of viscosity is destabilizing. If the free-stream velocity is subsonic, any laminar boundary-layer flow is unstable at sufficiently high (but finite) Reynolds numbers, whether or not the flow is stable in the inviscid fluid when only the inertia forces are considered.

The action of the inertia forces is more decisive for the stability of the laminar boundary layer if the free-stream velocity is supersonic. Because of the physical requirement that the relative phase velocity $(c - 1)$ of the boundary-layer disturbances must be subsonic, it follows that $c > 1 - \frac{1}{M_o} > 0$ and the quantity $\left[\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)\right]_{w=c}$ can be large enough negatively under certain conditions so that the stabilizing action of the inertia forces near the inner critical layer (where $w = c > 0$) is not overcome by the destabilizing action of viscosity near the solid surface. In that case, undamped disturbances cannot exist in the fluid, and the flow is stable at all values of the Reynolds number.

Regardless of the free-stream velocity, the distribution of the product of the density and the vorticity $\rho\frac{dw}{dy}$ across the boundary layer determines the actual limit of stability, or the minimum critical Reynolds number, for laminar boundary-layer flow in a viscous conductive gas (equation (9.8)). Since the distribution of $\rho\frac{dw}{dy}$ across the boundary layer in turn is determined by the free-stream Mach number and the thermal conditions at the solid surface, the effect of these physical parameters on the stability of laminar boundary-layer flow is readily evaluated.

b. Effect of Free-Stream Mach Number and Thermal Conditions at Solid Surface on Stability of Laminar Boundary Layer

The distribution of mean velocity and mean temperature (and therefore of $\rho\frac{dw}{dy}$) across the laminar boundary layer in a viscous conductive gas is strongly influenced by the fact that the viscosity of a gas increases with the temperature. (For most gases, $\mu \propto T^m$ ($m = 0.76$ for air) over a fairly wide temperature range.) When heat is transferred to the fluid through the solid surface, the temperature and viscosity near the surface both decrease along the outward normal, and the fluid near the surface is more retarded by the viscous shear than the fluid farther out from the surface - as compared with the isothermal Blasius flow. The velocity profile therefore always possesses a point of inflection (where $w'' = 0$) when heat is added to the fluid through the solid surface, provided there is no pressure gradient in the direction of the main flow. Since $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right) = \frac{w''}{T} - \frac{w'T'}{T^2}$, the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ vanishes and $\rho\frac{dw}{dy}$ has an extremum at some point in the fluid. On the other hand, if heat is withdrawn from the fluid through the solid surface, $\frac{\partial T}{\partial y}$ and $\frac{\partial \mu}{\partial y}$ are both positive near the surface and the fluid near the surface is less retarded than the fluid farther out - as compared with the Blasius flow. The velocity profile is therefore more convex near the surface than the Blasius profile.

As pointed out in section 11 of reference 8, the influence of the variable viscosity on the behavior of the product of the density and the vorticity $\rho\frac{dw}{dy}$ can be seen directly from the equations of motion for the mean flow. When there is no pressure gradient in the direction of the main flow, the fluid acceleration vanishes at the solid surface, or

$$\left(\frac{\partial\overline{\tau^*}}{\partial y^*}\right)_1 = \left[\frac{\partial}{\partial y^*}\left(\overline{\mu_1{}^*}\,\frac{\partial\overline{u^*}}{\partial y^*}\right)\right]_1 = 0 \tag{6.1}$$

and

$$\left(\frac{\partial^2\overline{u^*}}{\partial y^{*2}}\right)_1 = -\frac{1}{\overline{\mu_1{}^*}}\left(\frac{\partial\overline{\mu^*}}{\partial y^*}\right)_1\left(\frac{\partial\overline{u^*}}{\partial y^*}\right)_1 = -\frac{m}{T_1}\left(\frac{\partial\overline{T^*}}{\partial y^*}\right)_1\left(\frac{\partial\overline{u^*}}{\partial y^*}\right)_1 \tag{6.2}$$

Thus, when heat is added to the fluid through the solid surface $\left(T_1{}' < 0\right)$, $\left(\frac{\partial^2\overline{u^*}}{\partial y^{*2}}\right)_1$ is positive, and the velocity profile is concave near the surface and possesses a point of inflection for some value of $w > 0$; when heat is withdrawn from the fluid $\left(T_1{}' > 0\right)$, $\left(\frac{\partial^2\overline{u^*}}{\partial y^{*2}}\right)_1$ is negative, and the velocity profile is more convex near the surface than the Blasius profile.

The behavior of the quantity $\frac{\partial}{\partial y^*}\left(\frac{1}{\overline{T^*}}\,\frac{\partial\overline{u^*}}{\partial y^*}\right) = \frac{d}{dy^*}\left(\rho^*\frac{d\overline{u^*}}{dy^*}\right)$ is parallel to that of $\frac{\partial^2\overline{u^*}}{\partial y^{*2}}$. From equation (6.2), in nondimensional form,

$$\left[\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)\right]_1 = \left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_1 = -\frac{m+1}{T_1{}^2}T_1{}'w_1{}' \tag{6.3}$$

Differentiating the dynamic equations once and making use of the energy equation gives the following expression for $\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1$ (appendix D):

$$\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1 = \sigma(m+1)(\gamma-1)M_0{}^2\frac{(w_1{}')^3}{T_1{}^2} + 2(m+1)^2 w_1{}'\frac{(T_1{}')^2}{T_1{}^3} \tag{6.4}$$

Thus, for zero pressure gradient, $\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1$ is always positive. Now, if the surface is insulated, the quantity $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_1$ vanishes, but $\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1 > 0$ and $\frac{d}{dy}\left(\frac{w'}{T}\right)$ and $\frac{w'}{T}$ both increase with distance from the solid surface. Since $\frac{w'}{T} \longrightarrow 0$ far from the solid surface, $\frac{w'}{T}$ has a maximum and $\frac{d}{dy}\left(\frac{w'}{T}\right)$ vanishes for some value of $w > 0$. If heat is added to the fluid through the solid surface $\left(T_1{}' < 0\right)$, $\frac{d}{dy}\left(\frac{w'}{T}\right)$ is already positive at the surface, and since $\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1 > 0$, the quantity $\frac{d}{dy}\left(\frac{w'}{T}\right)$ vanishes at a point in the fluid which is farther from the surface than for an insulated boundary at the same Mach number (figs. 3(a) and (b)). Consequently, the value of $c = c_0$ for which the function

$(1 - 2\lambda)v(c) = - \pi(1 - 2\lambda) \frac{w_1'c}{T_1} \left[\frac{T^2}{(w')^3} \frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c}$ reaches the value 0.580 is larger than the value for the insulated surface. By equation (5.8), the effect of adding heat to the fluid through the solid surface is to reduce $R_{\theta_{cr_{min}}}$ and to destabilize the flow, as compared with the flow over an insulated surface at the same Mach number (fig. 6).

If heat is withdrawn from the fluid through the solid surface, $T_1' > 0$ and $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_1$ is negative. In fact, if the rate of heat transfer is sufficiently large, the quantity $\frac{d}{dy}\left(\frac{w'}{T}\right)$ does not vanish within the boundary layer (fig. 3(b)). The value of $c = c_o$ for which the function $(1 - 2\lambda)v(c)$ reaches the value 0.580 is smaller than for an insulated surface at the same Mach number, and by equation (5.8), the effect of withdrawing heat from the fluid through the solid surface is to increase $R_{\theta_{cr_{min}}}$ and to stabilize the flow, as compared with the flow over an insulated surface at the same Mach number (fig. 6). When the velocity of the free stream at the "edge" of the boundary layer is supersonic, the laminar boundary layer is completely stabilized if the rate at which heat is withdrawn through the solid surface reaches or exceeds a critical value that depends only on the Mach number, the Reynolds number, and the properties of the gas. The critical rate of heat transfer is that for which the quantity $\frac{d}{dy}\left(\frac{w'}{T}\right)$ is sufficiently large negatively near the surface (see equation (6.3)) so that $(1 - 2\lambda)v(c) = 0.580$ when $c = c_o = 1 - \frac{1}{M_o}$ (sections 4 and 5). Although detailed stability calculations for supersonic flow over a noninsulated surface have not been carried out, the function $(1 - 2\lambda)v(c)$ has been computed for noninsulated surfaces at $M_o = 1.30, 1.50, 2.00, 3.00$, and 5.00 by a rapid approximate method (appendix C). The corresponding estimated values of $R_{\theta_{cr_{min}}}$ were calculated from equation (5.8), and in figure 7 these values are plotted against T_1, the ratio of surface temperature (deg abs.) to free-stream temperature (deg abs.). At any given Mach number greater than unity the value of $R_{\theta_{cr_{min}}}$ increases rapidly as $c_o \longrightarrow 1 - \frac{1}{M_o}$; when c_o differs only slightly from $1 - \frac{1}{M_o}$, the stability of the laminar boundary layer is extremely sensitive to thermal conditions at the solid surface. At each value of $M_o > 1$, there is a critical value of the temperature ratio $T_{1_{cr}}$ for which $R_{\theta_{cr_{min}}} \longrightarrow \infty$. If $T_1 \leqq T_{1_{cr}}$, the laminar boundary layer is stable at all Reynolds numbers. The difference between the stagnation-temperature ratio and the critical-surface-temperature ratio, which is related to the heat-transfer coefficient, is plotted against Mach number in figure 8. Under free-flight conditions, for Mach numbers greater than some critical Mach number that depends largely on the altitude, the value of $T_s - T_{1_{cr}}$ is within the order of magnitude of the difference between stagnation temperature and surface temperature that actually exists because of heat radiation from the surface (references 14 and 15). In other words, the critical rate of heat withdrawal from the fluid for laminar stability is within the order of magnitude of the calculated rate of heat conduction through the solid surface which balances the heat radiated from the surface under equilibrium conditions. The calculations in appendix E show that this critical Mach number is approximately 3 at 50,000 feet altitude and approximately 2 at 100,000 feet altitude. Thus, for $M_o > 3$ (approx.) at 50,000 feet altitude and $M_o > 2$ (approx.) at 100,000 feet altitude, the laminar boundary-layer flow for thermal equilibrium is completely stable in the absence of an adverse pressure gradient in the free stream.

When there is actually no heat conduction through the solid surface, the limit of stability of the laminar boundary layer depends only on the free-stream Mach number, that is, on the extent of the "aerodynamic heating" $\left(\text{of the order of } \overline{u_1^*}\left(\frac{\partial \overline{u^*}}{\partial y^*}\right)^2\right)$ near the solid surface. A good indication of the influence of the free-stream Mach number on the distribution of $\rho\frac{dw}{dy}$ across the boundary layer for an insulated surface is obtained from a rough estimate of the location of the point at which $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ reaches a positive maximum $\left(\text{or } \frac{d^2}{dy^2}\left(\rho\frac{dw}{dy}\right) \text{ vanishes}\right)$. Differentiating the dynamic equations of mean motion twice and making use of the energy and continuity equations yields the following result for an insulated surface:

$$\left[\frac{d^3}{dy^3}\left(\frac{w'}{T}\right)\right]_1 = -\frac{b^2}{2}\frac{\left(w_1'\right)^2}{T_1^{m+2}} \tag{6.5}$$

where $b = \delta\sqrt{\frac{\overline{u_o^*}}{\overline{\upsilon_o^*}\ x^*}}$. From equations (6.4) and (6.5) the value of c at which $\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)$ vanishes, or $\frac{d}{dy}\left(\frac{w'}{T}\right)$ reaches a maximum, is given roughly for air by

$$c \approx \frac{w_1'\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1}{-\left[\frac{d^3}{dy^3}\left(\frac{w'}{T}\right)\right]_1} \approx \frac{M_o^{\ 2}}{T_1^{\ 2-m}} = \frac{M_o^{\ 2}}{\left(1 + 0.2025M_o^{\ 2}\right)^{1.24}} \tag{6.6}$$

in which $w_1' \approx \frac{b(0.3320)}{T_1}$ (appendix B). In other words, the point in the fluid at which $\frac{d}{dy}\left(\frac{w'}{T}\right)$ attains a maximum moves farther out from the surface as the Mach number is increased - at least in the range $0 \leqq M_o \leqq 4.5$ (approx.); therefore the value of c for which $\frac{d}{dy}\left(\frac{w'}{T}\right)$ vanishes and the value of $c = c_o$ for which $(1 - 2\lambda)v(c)$ reaches the value 0.580 both increase with the Mach number (fig. 3(a)). By equation (5.8), the value of $R_{\theta_{cr_{min}}}$ for the laminar boundary-layer flow over an insulated surface decreases as the Mach number increases and the flow is destabilized, as compared with the Blasius flow (fig. 5).

c. Results of Detailed Stability Calculations for Insulated and Noninsulated Surfaces

From the results of the detailed stability calculations for several representative cases (figs. 4 to 6), a quantitative estimate can be made of the effect of free-stream Mach number and thermal conditions at the solid surface on the stability of laminar boundary-layer flow. For the insulated surface, the value of $R_{\theta_{cr_{min}}}$ is 92 when $M_o = 1.30$ as compared with a value of 150 for the Blasius flow. For the noninsulated surface at $M_o = 0.70$, the value of $R_{\theta_{cr_{min}}}$ is 63 when $T_1 = 1.25$ (heat added to fluid), $R_{\theta_{cr_{min}}} = 126$ when $T_1 = 1.10$ (insulated surface), and $R_{\theta_{cr_{min}}} = 5150$ when $T_1 = 0.70$ (heat withdrawn from fluid). Since $R_{x*} \approx 2.25 R_\theta^2$, (the value of $\theta\sqrt{\frac{u_o^*}{\nu_o^*\, x^*}}$, which is proportional to the skin-friction coefficient, differs only slightly from the Blasius value of 0.6667) the effect of the thermal conditions at the solid surface on R_{x*} is even more pronounced. The value of R_{x*} is 60×10^6 when $T_1 = 0.70$ and $M_o = 0.70$, as compared with a value of 51×10^3 for the Blasius flow $\left(T_1 = 1 \text{ and } M_o = 0\right)$. For the insulated surface the value of $R_{x*_{cr_{min}}}$ declines from the Blasius value for $M_o = 0$ to a value of 19×10^3 at $M_o = 1.30$. The extreme sensitivity of the limit of stability of the laminar boundary layer to thermal conditions at the solid surface when $T_1 < 1$ is accounted for by the fact that c_o is small when $T_1 < 1$ and $M_o < 1$ (or M_o is not much greater than unity) and $R_{\theta_{cr_{min}}} \approx \frac{1}{c_o^{\,4}}$ (equation (5.8)). Small changes in c_o, therefore, produce large changes in $R_{\theta_{cr_{min}}}$. In addition, when $T_1 < 1$, small changes in the thermal conditions at the solid surface produce appreciable changes in $\frac{d}{dy}\left(\frac{w'}{T}\right)$ (equation (6.3)) and, therefore, in the value of c_o.

Not only is the value of $R_{\theta_{cr_{min}}}$ affected by the thermal conditions at the solid surface and by the free-stream Mach number but the entire curve of α_θ against R_θ for the neutral disturbances is also affected. (See figs. 4(k) and 4(l).) When the surface is insulated (and $M_o \neq 0$), or heat is added to the fluid $\left(T_1 = 1.25\right)$, $\alpha_\theta \longrightarrow \alpha_s \neq 0$ as $R_\theta \longrightarrow \infty$ along the upper branch of the curve of neutral stability. In other words, there is a finite range of unstable wave lengths even in the limiting case of an infinite Reynolds number (inviscid fluid). However, $\alpha \longrightarrow 0$ as $R_\theta \longrightarrow \infty$ for the Blasius flow, or when heat is withdrawn from the fluid. This behavior is in complete agreement with the results obtained in section 4 and in reference 8.

A comparison between the curves of α_θ against R_θ for $T_1 = 1.25$ and $T_1 = 0.70$ at $M_o = 0.70$ shows that withdrawing heat from the fluid not only stabilizes the flow by increasing $R_{\theta_{cr_{min}}}$ but also greatly reduces the range of unstable wave numbers (α_θ). On the other hand, the addition of heat to the fluid through the solid surface greatly increases the range of unstable wave numbers.

It should also be noted that for given values of α_θ, c, and R_θ the time frequencies of the boundary-layer disturbances in the high-speed flow of a gas are considerably greater than the frequencies of the familiar Tollmien waves observed in low-speed flow. The actual time frequency n^* expressed nondimensionally is as follows:

$$\frac{n^* \, \overline{\nu_o^*}}{\left(\overline{u_o^*}\right)^2} = \frac{c\alpha_\theta}{2\pi R_\theta}$$

For given values of c, α_θ, and R_θ the frequency increases as the square of the free-stream velocity.

d. Instability of Laminar Boundary Layer and Transition to Turbulent Flow

The value of $R_{\theta_{cr_{min}}}$ obtained from the stability analysis for a given laminar boundary-layer flow is the value of the Reynolds number at which self-excited disturbances first appear in the boundary layer. As Prandtl (reference 12) carefully pointed out, these initial disturbances are not turbulence, in any sense, but slowly growing oscillations. The value of the Reynolds number at which boundary-layer disturbances propagated along the surface will be amplified to a sufficient extent to cause turbulence must be larger than $R_{\theta_{cr_{min}}}$ in any case; for the insulated flat-plate flow at low speeds and with no pressure gradient, the transition Reynolds number $R_{\theta_{tr}}$ is found to be three to seven times as large as the value of $R_{\theta_{cr_{min}}}$ (references 6 and 7). The value of $R_{\theta_{tr}}$ depends not only on $R_{\theta_{cr_{min}}}$ but also on the initial magnitude of the disturbances with the most "dangerous" frequencies (those with greatest amplification), on the rate of amplification of these disturbances, and on the physical process (as yet unknown) by which the quasi-stationary laminar flow is finally destroyed by the amplified oscillations. (See, for example, references 16 and 17.) The results of the stability analysis nevertheless permit certain general statements to be made concerning the effect of free-stream Mach number and thermal conditions at the solid surface on transition. The basis for these statements is summarized as follows:

(1) In many problems of technical interest in aeronautics the level of free-stream turbulence (magnitude of initial disturbances) is sufficiently low so that the origin of transition is always to be found in the instability of the laminar boundary layer. In other words, the value of $R_{\theta_{cr_{min}}}$ is an absolute lower limit for transition.

(2) The effect of the free-stream Mach number and the thermal conditions at the solid surface on the stability limit $\left(R_{\theta_{cr_{min}}}\right)$ is overwhelming. For example, for $M_0 = 0.70$, the value of $R_{\theta_{cr_{min}}}$ when $T_1 = 0.70$ (heat withdrawn from fluid) is more than 80 times as great as the value of $R_{\theta_{cr_{min}}}$ when $T_1 = 1.25$ (heat added to fluid).

(3) The maximum rate of amplification of the self-excited boundary-layer disturbances propagated along the surface varies roughly as $1/\sqrt{R_{\theta_{cr_{min}}}}$. (This approximation agrees closely with the numerical results obtained by Pretsch (reference 18) for the case of an incompressible fluid.) The effect of withdrawing heat from the fluid, for example, is not only to increase $R_{\theta_{cr_{min}}}$ and

stabilize the flow in that manner but also to decrease the initial rate of amplification of the unstable disturbances. In other words, for a given level of free-stream turbulence, the interval between the first appearance of self-excited disturbances and the onset of transition is expected to be much longer for a relatively stable flow, for which $R_{\theta_{cr_{min}}}$ is large, than for a relatively unstable flow, for which $R_{\theta_{cr_{min}}}$ is small and the initial rate of amplification is large.

On the basis of these observations, transition is delayed $\left(R_{\theta_{tr}}\right.$ increased$\left.\right)$ by withdrawing heat from the fluid through the solid surface and is advanced by adding heat to the fluid through the solid surface, as compared with the insulated surface at the same Mach number. For the insulated surface, transition occurs earlier as the Mach number is increased, as compared with the flat-plate flow at very low Mach numbers. When the free-stream velocity at the edge of the boundary layer is supersonic, transition never occurs if the rate of heat withdrawal from the fluid through the solid surface reaches or exceeds a critical value that depends only on the Mach number (section 6b and figs. 7 and 8).

A comparison between the results of the present analysis and measurements of transition is possible only when the free-stream pressure gradient is zero or is held fixed while the free-stream Mach number or the thermal conditions at the solid surface are varied. Liepmann and Fila (reference 19) have measured the movement of the transition point on a flat plate at a very low free-stream velocity when heat is applied to the surface. They found by means of the hot-wire anemometer that $R_{x^*_{tr}}$ declined from 5×10^5 for the insulated surface to a value of approximately 2×10^5 for $T_1 = 1.36$ when the level of free-stream turbulence $\sqrt{\frac{\overline{(u^{*\prime})^2}}{(\overline{u_o^*})^2}}$ was 0.17 percent, or to a value of 3×10^5 when $\sqrt{\frac{\overline{(u^{*\prime})^2}}{(\overline{u_o^*})^2}} = 0.05$ percent and $T_1 = 1.40$. The value of $R_{\theta_{tr}}$ declines from 470 (approx.) to 300 (approx.) in the first case and to 365 in the second.

Frick and McCullough (reference 20) observed the variation in the transition Reynolds number when heat is applied to the upper surface of an NACA 65,2-016 airfoil at the nose section alone, at the section just ahead of the minimum pressure station, and for the entire laminar run. When heat is applied only to the nose section, the transition Reynolds number (determined by total-pressure-tube measurements) was practically unchanged. Near the nose, $R_\theta << R_{\theta_{cr_{min}}}$ and the strong favorable pressure gradient in the region of the stagnation point stabilizes the laminar boundary layer to such an extent that the addition of heat to the fluid has only a negligible effect. When heat is applied, however, to the section just ahead of the minimum pressure point, where the pressure gradients are moderate, the transition Reynolds number $R_{\theta_{tr}}$ declined to a value of 1190 for $T_1 \approx 1.14$, compared with a value of 1600 for the insulated surface. When heat is applied to the entire laminar run, $R_{\theta_{tr}}$ declined to a value of 1070 for $T_1 \approx 1.14$.

It would be interesting to investigate experimentally the stabilizing effect of a withdrawal of heat from the fluid at supersonic velocities. At any rate, on the basis of the results obtained in the experimental investigations of the effect of heating on transition at low speeds, the results of the stability analysis give the proper direction of this effect.

7. Stability of the Laminar Boundary-Layer Flow of a Gas with a Pressure Gradient in the Direction of the Free Stream

For the case of an incompressible fluid, Pretsch (reference 9) has shown that even with a pressure gradient in the direction of the free stream, the local mean-velocity distribution across the boundary layer completely determines the stability characteristics of the local laminar boundary-layer flow at large Reynolds numbers. From physical considerations this statement should apply also to the compressible fluid, provided only the stability of the flow in the boundary layer is considered and not the possible interaction of the boundary layer and the main "external" flow. Further study is required to settle this question.

If only the local mean velocity-temperature distribution across the boundary layer is found to be significant for laminar stability in a compressible fluid, the criterions obtained in the present paper and in reference 8 are then immediately applicable to laminar boundary-layer gas flows in which there is a free-stream pressure gradient. The quantitative effect of a pressure gradient on laminar stability could be readily determined by means of the approximate estimate of $R_{\theta_{cr_{min}}}$ (equation (5.7)), in terms of the distribution of the quantity $\rho \frac{dw}{dy}$ across the boundary layer. Such calculations (unpublished) have already been carried out by Dr. C. C. Lin of Brown University for the incompressible fluid by means of the approximate estimate of $R_{\delta^*_{cr_{min}}}$ given in reference 5, part III.

In any event, the qualitative effect of a free-stream pressure gradient on the local distribution of $\rho \frac{dw}{dy}$ across the boundary layer is evidently the same in a compressible fluid as in an incompressible fluid. If the effect of the local pressure gradient alone is considered, the velocity distribution across the boundary layer is "fuller" or more convex for accelerated than for uniform flow, and conversely, less convex for decelerated flow. Thus, from the results of the present paper the effect of a negative pressure gradient on the laminar boundary-layer flow of gas is stabilizing, so far as the local mean velocity-temperature distribution is concerned, while a positive pressure gradient is destabilizing. For the incompressible fluid, this fact is well established by the Rayleigh-Tollmien criterion (reference 3), the work of Heisenberg (reference 1) and Lin (reference 5), and a mass of detailed calculations of stability limits from the curves of α against R for the neutral disturbances. These calculations were recently carried out by several German investigators for a comprehensive series of pressure gradient profiles. (See, for example, references 9 and 21.)

Some idea of the relative influence on laminar stability of the thermal conditions at the solid surface and the free-stream pressure gradient is obtained from the equations of mean motion. At the surface,

$$\left(\frac{\partial \overline{\tau^*}}{\partial y^*}\right)_1 = \left[\frac{\partial}{\partial y^*}\left(\overline{\mu_1^*}\,\frac{\partial \overline{u^*}}{\partial y^*}\right)\right]_1 = \frac{d\overline{p^*}}{dx^*} = -\overline{\rho_0^*}\,\overline{u_0^*}\,\frac{d\overline{u_0^*}}{dx^*} \qquad (7.1)$$

or

$$\left[\frac{d}{dy}\left(\rho\,\frac{dw}{dy}\right)\right]_1 = -\frac{m+1}{T_1^{\,2}}\,T_1{}'w_1{}' - \frac{1}{T_1^{\,m+1}}\,\frac{\delta^2}{\overline{\upsilon_0^*}}\,\frac{d\overline{u_0^*}}{dx^*} \qquad (7.2)$$

In a region of small or moderate pressure gradients $\left(\left|\frac{\delta^2}{\nu_o^*}\frac{d\overline{u_o^*}}{dx}\right| \leq 2,\right.$ say$\Big)$ the distribution of $\rho\frac{dw}{dy}$ is sensitive to the thermal conditions at the solid surface. For example, the chordwise position of the point of instability of the laminar boundary layer on an airfoil with a flat pressure distribution is expected to be strongly influenced by heat conduction through the surface. (See reference 20.) For the insulated surface, the equations of mean motion yield the following relation (appendix D), which does not involve the pressure gradient explicitly:

$$\left[\frac{d^2}{dy^2}\left(\rho\frac{dw}{dy}\right)\right]_1 = \sigma(m + 1)(\gamma - 1)M_o^2\frac{\left(w_1'\right)^3}{T_1^2} > 0 \tag{7.3}$$

The effect of "aerodynamic heating" at the surface opposes the effect of a favorable pressure gradient so far as the distribution of $\rho\frac{dw}{dy}$ across the boundary layer is concerned (equations (7.2) and (7.3)). The relative quantitative influence of these two effects on laminar stability can only be settled by actual calculations of the laminar boundary-layer flow in a compressible fluid with a free-stream pressure gradient. A method for the calculation of such flows over an insulated surface is given in reference 22.

When the local free-stream velocity at the edge of the boundary layer is supersonic, a negative pressure gradient can have a decisive effect on laminar stability. The local laminar boundary-layer flow over an insulated surface, for example, is expected to be completely stable when the magnitude of the local negative pressure gradient reaches or exceeds a critical value that depends only on the local Mach number and the properties of the gas. The critical magnitude of the pressure gradient is that which makes the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ sufficiently large negatively near the surface so that

$$-\left[1 - 2\lambda(c)\right]\pi\frac{w_1'c}{T_1}\left[\frac{T^2}{(w')^3}\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c} = 0.580$$

when $c = 1 - \frac{1}{M_o}$.

It has already been shown in the present paper that when $M_0 > 3$ (approx.) the laminar boundary-layer flow with a uniform free-stream velocity is completely stable under free-flight conditions when the solid surface is in thermal equilibrium, that is, when the heat conducted from the fluid to the surface balances the heat radiated from the surface (section 6b). The laminar boundary-layer flow for thermal equilibrium should be completely stable for $M_0 > M_s$, say, where $M_s < 3$ if there is a negative pressure gradient in the direction of the free stream. Favorable pressure gradients exist over the forward part of sharp-nosed airfoils and bodies of revolution moving at supersonic velocities, and the limits of stability $\left(R_{\theta_{cr_{min}}}\right)$ of the laminar boundary layer should be calculated in such cases.

CONCLUSIONS

From a study of the stability of the laminar boundary layer in a compressible fluid, the following conclusions were reached:

1. In the compressible fluid as in the incompressible fluid, the influence of viscosity on the laminar boundary-layer flow of a gas is destabilizing at very large Reynolds numbers. If the free-stream velocity is subsonic, any laminar boundary-layer flow of gas is unstable at sufficiently high Reynolds numbers.

2. Regardless of the free-stream Mach number, if the product of the mean density and the mean vorticity has an extremum $\left(\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)\right.$ vanishes$\left.\right)$ for some value of $w > 1 - \frac{1}{M_0}$ (where w is the ratio of mean velocity component parallel to the surface to the free-stream velocity, and where M_0 is the free-stream Mach number) the flow is unstable at sufficiently high Reynolds numbers.

3. The actual limit of stability of laminar boundary-layer flow, or the minimum critical Reynolds number $R_{\theta_{cr_{min}}}$, is determined largely by the distribution of the product of the mean density and the mean vorticity across the boundary layer. An approximate estimate of $R_{\theta_{cr_{min}}}$ is obtained that serves as a criterion for

the influence of free-stream Mach number and thermal conditions at the solid surface on laminar stability. For zero pressure gradient, this estimate reads as follows:

$$R_{\theta_{cr_{min}}} \approx \frac{6}{T_1} \frac{\left[T\left(c_o\right)\right]^{1.76}}{c_o^{\,4}\sqrt{1 - M_o^{\,2}\left(1 - c_o\right)^2}}$$

where T is the ratio of temperature at a point within the boundary layer to free-stream temperature, T_1 is the ratio of temperature at the solid surface to the free-stream velocity, and c_o is the value of c (the ratio of phase velocity of disturbance to the free-stream velocity) for which $(1 - 2\lambda)v = 0.580$. The functions $v(c)$ and $\lambda(c)$ are defined as follows:

$$v(c) = \frac{-\pi\left(\frac{\partial w}{\partial \eta}\right)_1 c}{T_1}\left[\frac{T^2}{\left(\frac{\partial w}{\partial \eta}\right)^3}\frac{\partial}{\partial \eta}\left(\frac{1}{T}\frac{\partial w}{\partial \eta}\right)\right]_{w=c}$$

$$\lambda(c) = \frac{\eta\left(\frac{\partial w}{\partial \eta}\right)_1}{c} - 1$$

where

η nondimensional distance from surface

4. On the basis of the stability criterion in conclusion 3 and a study of the equations of mean motion, the effect of adding heat to the fluid through the solid surface is to reduce $R_{\theta_{cr_{min}}}$ and to

destabilize the flow, as compared with the flow over an insulated surface at the same Mach number. Withdrawing heat through the solid surface has exactly the opposite effect. The value of $R_{\theta_{cr_{min}}}$ for the laminar boundary-layer flow over an insulated surface decreases as the Mach number increases, and the flow is destabilized, as compared with the Blasius flow at low speeds.

5. When the free-stream velocity is supersonic, the laminar boundary layer is completely stabilized if the rate at which heat is withdrawn from the fluid through the solid surface reaches or exceeds a certain critical value. The critical rate of heat transfer, for which $R_{\theta_{cr_{min}}} \longrightarrow \infty$, is that which makes the quantity $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ sufficiently large negatively near the surface so that $\left[1 - 2\lambda(c)\right] v(c) = 0.580$ when $c = c_o = 1 - \frac{1}{M_o}$. Calculations for several supersonic Mach numbers between 1.30 and 5.00 show that for $M_o > 3$ (approx.) the critical rate of heat withdrawal for laminar stability is within the order of magnitude of the calculated rate of heat conduction through the solid surface that balances the heat radiated from the surface under free-flight conditions. Thus, for $M_o > 3$ (approx.) the laminar boundary-layer flow for thermal equilibrium is completely stable at all Reynolds numbers in the absence of a positive (adverse) pressure gradient in the direction of the free stream.

6. Detailed calculations of the curves of wave number (inverse wave length) against Reynolds number for the neutral boundary-layer disturbances for 10 representative cases of insulated and non-insulated surfaces show that also at subsonic speeds the quantitative effect on stability of the thermal conditions at the solid surface is very large. For example, at a Mach number of 0.70, the value of $R_{\theta_{cr_{min}}}$ is 63 when $T_1 = 1.25$ (heat added to fluid), $R_{\theta_{cr_{min}}} = 126$ when $T_1 = 1.10$ (insulated surface), and $R_{\theta_{cr_{min}}} = 5150$ when $T_1 = 0.70$ (heat withdrawn from fluid). Since $R_x* \approx 2.25R_\theta{}^2$, the effect on $R_{x*_{cr_{min}}}$ is even greater.

7. The results of the analysis of the stability of laminar boundary-layer flow by the linearized method of small perturbations must be applied with care to predictions of transition, which is a nonlinear phenomenon of a different order. Withdrawing heat from the

fluid through the solid surface, however, not only increases $R_{\theta_{cr_{min}}}$ but decreases the initial rate of amplification of the self-excited disturbances, which is roughly proportional to $1/\sqrt{R_{\theta_{cr_{min}}}}$; addition of heat to the fluid through the solid surface has the opposite effect. Thus, it can be concluded that (a) transition is delayed $\left(R_{\theta_{tr}}\right.$ increased$\left.\right)$ by withdrawing heat from the fluid and advanced by adding heat to the fluid through the solid surface, as compared with the insulated surface at the same Mach number, (b) for the insulated surface, transition occurs earlier as the Mach number is increased, (c) when the free stream velocity is supersonic, transition never occurs if the rate of heat withdrawal from the fluid through the solid surface reaches or exceeds the critical value for which $R_{\theta_{cr_{min}}} \longrightarrow \infty$. (See conclusion 5.)

Unlike laminar instability, transition to turbulent flow in the boundary layer is not a purely local phenomenon but depends on the previous history of the flow. The quantitative effect of thermal conditions at the solid surface on transition depends on the existing pressure gradient in the direction of the free stream, on the part of the solid surface to which heat is applied, and so forth, as well as on the initial magnitude of the disturbances (level of free-stream turbulence).

A comparison between conclusion 7(a), based on the results of the stability analysis, and experimental investigations of the effect of surface heating on transition at low speeds shows that the results of the present paper give the proper direction of this effect.

8. The results of the present study of laminar stability can be extended to include laminar boundary-layer flows of a gas in which there is a pressure gradient in the direction of the free stream. Although further study is required, it is presumed that only the local mean velocity-temperature distribution determines the stability of the local boundary-layer flow. If that should be the case, the effect of a pressure gradient on laminar stability could be easily calculated through its effect on the local distribution of the product of mean density and mean vorticity across the boundary layer.

When the free-stream velocity at the "edge" of the boundary layer is supersonic, by analogy with the stabilizing effect of a withdrawal of heat from the fluid, it is expected that the laminar boundary-layer flow is completely stable at all Reynolds numbers when the negative (favorable) pressure gradient reaches or exceeds a certain critical value that depends only on the Mach number and the properties of the gas. The laminar boundary-layer flow over a surface in thermal equilibrium should be completely stable for $M_o > M_s$, say, where $M_s < 3$ if there is a negative pressure gradient in the direction of the free stream.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., September 5, 1946

APPENDIX A

CALCULATION OF INTEGRALS APPEARING IN THE INVISCID SOLUTIONS

In order to calculate the limits of stability of the laminar boundary layer from relations (2.21) to (2.29) between the values of phase velocity, wave number, and Reynolds number, it is first necessary to calculate the values of the integrals K_1, H_1, H_2, N_2, M_3, N_3, and so forth, which appear in the expressions for the inviscid solutions $\varphi_1(y)$ and $\varphi_2(y)$ and their derivatives at the edge of the boundary layer. These integrals are as follows (equations (2.13), (2.9), and (2.10)):

$$H_1(c) = \int_{y_1}^{y_2} \frac{(w - c)^2}{T}\, dy$$

$$K_1(c) = \int_{y_1}^{y_2} \frac{T - M_o^2(w - c)^2}{(w - c)^2}\, dy$$

$$N_2(c) = K_1H_1 - K_2 = \int_{y_1}^{y_2} \frac{T - M_o^2(w - c)^2}{(w - c)^2}\, dy \int_{y_1}^{y} \frac{(w - c)^2}{T}\, dy = H_2(c)$$

$$M_3(c) = H_2H_1 - H_3 = \int_{y_1}^{y_2} \frac{(w - c)^2}{T}\, dy \int_{y}^{y_2} \frac{T - M_o^2(w - c)^2}{(w - c)^2}\, dy \int_{y_1}^{y} \frac{(w - c)^2}{T}\, dy$$

$N_3(c) = K_1H_2 - K_3$

$$= \int_{y_1}^{y_2} \frac{T - M_o^2(w - c)^2}{(w - c)^2}\,dy \int_{y_1}^{y} \frac{(w - c)^2}{T}\,dy \int_{y}^{y_2} \frac{T - M_o^2(w - c)^2}{(w - c)^2}\,dy$$

and so forth.

Terms of higher order than α^3 in the series expressions for φ_1 and φ_2 are neglected. When $\alpha < 1$, the error involved is small because the terms in the series decline like $\frac{\alpha^n}{n!}$. Even for $\alpha > 1$, however, this approximation is justified, at least for the values of c that appear in the stability calculations for the 10 representative cases selected in the present paper. For example, the leading term in R.P. $N_{2k+1}(c)$, where $k = 2,3, \ldots$, is approximately $\frac{1}{k!}\left[\frac{c^3}{3(1 - c)}\right]^{k-1}$ multiplied by the leading term in R.P. $N_3(c)$. The quantity in the brackets is at most 0.12 in the present calculations; for example, R.P. $N_5(c) \approx 0.06$ R.P. $N_3(c)$. Moreover, R.P. $N_{2k}(c) \approx (1 - c)$ R.P. $N_{2k+1}(c)$. Similar approximate relations exist between R.P. $M_{2k}(c)$ and R.P. $M_3(c)$; and, in addition, R.P. $M_3(c) \approx (1 - c)\frac{c^3}{6}$ R.P. $N_3(c) \approx 0.015$ R.P. $N_3(c)$, at most.

The only integral for which the imaginary part is calculated is $K_1(c)$. At the end of this appendix, it is shown that the contributions of the imaginary parts of H_2, M_3, and N_3 are negligible in comparison with the contribution of I.P. $K_1(c)$.

General Plan of Calculation

The method of calculation adopted must take into account the fact that the value of $\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)$ at the point $y = y_c$, where $w = c$, strongly influences the stability of the laminar boundary layer. Accordingly, the integrals are broken into two parts; for example,

$$K_1(c) = \int_{y_1}^{y_j} \frac{T}{(w - c)^2}\, dy + \int_{y_j}^{y_2} \frac{T}{(w - c)^2}\, dy - M_o^2$$

$$= K_{11}(c) + K_{12}(c) - M_o^2$$

where $y_j > y_c$. The integral $K_{11}(c)$, which involves $\left[\frac{d}{dy}\left(\rho\frac{dw}{dy}\right)\right]_{w=c}$, is calculated very accurately, whereas $K_{12}(c)$ is calculated by a more approximate method as follows:

$$K_{12}(c) = \int_{y_j}^{y_2} \frac{T}{(w - c)^2}\, dy \qquad (1)$$

This integral is evaluated as a power series in c. The velocity profile $w(y)$ is approximated by a parabolic arc plus a straight-line segment for purposes of integration. In the more complex integrals H_2, M_3, and N_3, the indefinite integrals $\int_{y_j}^{y} \frac{T}{(w - c)^2}\, dy$ and $\int_{y}^{y_2} \frac{T}{(w - c)^2}\, dy$ are evaluated by 21 or 41 point numerical integration by means of Simpson's rule. The values of $w(y)$ are read from the velocity profiles of figures 1 and 2. The value of $y_j - y_1 = a$ is 0.40 in the present series of calculations; this value is chosen so that the point $y = y_j$ is never too close to the singularity at $y = y_c$. Take

$$K_{11}(c) = \int_{y_1}^{y_j} \frac{T}{(w - c)^2}\, dy \qquad (2)$$

The integral $K_{11}(c)$, or the indefinite integral $\int_{y_1}^{y} \frac{T}{(w - c)^2} dy$ that appears in H_2, M_3, and N_3, is evaluated by expanding the integrand in a Taylor's series in $y - y_c$ and then integrating the series term by term. The path of integration must be taken below the point $y = y_c$ in the complex y-plane.

Instead of calculating the values of the velocity and temperature derivatives $w_c^{(n)}$ and $T_c^{(n)}$ directly, it is simpler to relate these derivatives to their values at the surface by Taylor's series of the form

$$w_c^{(n)} = w_1^{(n)} + w_1^{(n+1)} \left(y_c - y_1\right) + \frac{w_1^{(n+2)}}{2!} \left(y_c - y_1\right)^2 + \ldots$$

The derivatives at the surface $w_1^{(n)}$ and $T_1^{(n)}$ are calculated from the equations of mean motion (appendix B).

The integral $K_{11}(c)$, for example, is finally obtained as a power series in $y_c - y_1 = \sigma$ and in $y_j - y_c = a - \sigma$, plus terms involving $\log \sigma$. The phase velocity c is related to σ by

$$c = w_1' \left(\sigma + \frac{A_2}{2} \sigma^2 + \frac{A_3}{3!} \sigma^3 + \ldots\right)$$

where

$$A_k = \frac{w_1^{(k)}}{w_1'}$$

Terms up to the order of a^5 are retained in order to include all terms involving w_1^{vii}.

Detailed Calculations

In order to illustrate the method, the evaluation of $K_1(c)$ is given in some detail, as follows:

(1) Evaluation of $K_1(c)$:

$$K_1(c) = \int_{y_1}^{y_2} \frac{T}{(w - c)^2}\, dy - M_0^2$$

(a) Define

$$K_{11}(c) = \int_{y_1}^{y_3} \frac{T}{(w - c)^2}\, dy$$

Now

$$\frac{T}{(w - c)^2} = \frac{T}{\left(w_c'\right)^2 \left(y - y_c\right)^2 \psi^2}$$

where

$$\psi(y) = 1 + \frac{w_c''}{2w_c'}\left(y - y_c\right) + \frac{w_c'''}{3!w_c'}\left(y - y_c\right)^2 + \ldots$$

The function $\frac{T}{\psi^2}$ is developed in a Taylor's series around the point $w = c$ as follows:

$$\frac{T}{\psi^2} = \left(\frac{T}{\psi^2}\right)_{y=y_c} + \left(\frac{T}{\psi^2}\right)_c' \left(y - y_c\right) + \left(\frac{T}{\psi^2}\right)_c'' \frac{\left(y - y_c\right)^2}{2!} + \ldots$$

where

$$\psi_c = 1$$

$$\psi_c' = \frac{w_c''}{2w_c'}$$

$$\psi_c^{(k)} = \frac{w_c^{(k+1)}}{(k+1)w_c'}$$

Then

$$K_{11}(c) = \frac{1}{(w_c')^2}\int_{y_1-y_c}^{y_j-y_c} \frac{d(y-y_c)}{(y-y_c)^2}\left[\left(\frac{T}{\psi^2}\right)_c + \left(\frac{T}{\psi^2}\right)_c'(y-y_c) + \left(\frac{T}{\psi^2}\right)_c''\frac{(y-y_c)^2}{2!} + \ldots\right]$$

and

$$K_{11}(c) = \frac{1}{(w_c')^2}\left\{\left[-\frac{T_c}{y-y_c}\right]_{y_1}^{y_j} + \left(\frac{T}{\psi^2}\right)_c' \ln\left(\frac{y_j-y_c}{y_1-y_c}\right) + \frac{1}{2}\left(\frac{T}{\psi^2}\right)_c''(y_j-y_1)\right.$$

$$+ \frac{1}{12}\left(\frac{T}{\psi^2}\right)_c'''\left[(y_j-y_c)^2 - (y_1-y_c)^2\right] + \ldots$$

$$\left. + \frac{1}{(k)(k+1!)}\left(\frac{T}{\psi^2}\right)_c^{(k+1)}\left[(y_j-y_c)^k - (y_1-y_c)^k\right] + \ldots\right\}$$

where

$$y_1 - y_c = \left|y_1 - y_c\right| e^{-i\pi}$$

$$y_j - y_c = \left(y_j - y_1\right) - \left(y_c - y_1\right) = a - \sigma$$

$$\sigma = y_c - y_1$$

The coefficients $\left(\frac{T}{\psi^2}\right)^{(k)}$ are expressed in terms of derivatives of T and w at $y = y_1$ as follows:

Define

$$f_k(y) = \frac{1}{(k-1)\ k!}\ \frac{1}{(w')^2}\left(\frac{T}{\psi^2}\right)^k \qquad k \geqq 2$$

$$f_o(y) = -\frac{T}{(w')^2}$$

$$f_1(y) = \frac{1}{(w')^2}\left(\frac{T}{\psi^2}\right)' = -\frac{T^2}{(w')^3}\frac{d}{dy}\left(\frac{w'}{T}\right)$$

Then

$$f_k\left(y_c\right) = \frac{1}{\left(w_c'\right)^2(k-1)k!}\left[\left(\frac{T}{\psi^2}\right)^k\right]_{y_c}$$

$$= f_k\left(y_1\right) + f_k'\left(y_1\right)\left(y_c - y_1\right) + \frac{f_k''\left(y_1\right)}{2!}\left(y_c - y_1\right)^2 + \ldots$$

(The method adopted for the calculation of $f_k^{(n)}(y_1)$ from the velocity and temperature derivatives $w_1^{(j)}$ and $T_1^{(j)}$ is given at the end of this appendix.)

From the expression for $K_{11}(c)$,

$$\begin{aligned} \text{I.P. } K_{11}(c) &= \text{I.P. } K_1(c) \\ &= \pi f_1(y_c) \\ &= \pi\left[f_1(y_1) + \sigma f_1'(y_1) + \cdots + \frac{\sigma^5}{5!} f_1^{v}(y_1)\right] \end{aligned}$$

and

$$\begin{aligned} \text{R.P. } K_{11}(c) + \frac{T_1}{w_1'c} = {} & c_0 + c_1\sigma + c_2\sigma^2 + \cdots + c_5\sigma^5 \\ & + \frac{\text{I.P. } K_{11}(c)}{\pi} \ln\left(\frac{a-\sigma}{\sigma}\right) + \frac{1}{a-\sigma}\left[f_o(y_1)\right. \\ & + \sigma f_o'(y_1) + \cdots + \frac{\sigma^k f_o^{(k)}(y_1)}{k!} + \cdots \\ & \left. + \frac{\sigma^6 f_o^{vi}(y_1)}{720}\right] \end{aligned}$$

where

$$\sigma = y_c - y_1$$

$$c_k = s_k + \frac{f_o^{(k+1)}(y_1)}{(k+1)!} - d_{k+1} f_o(y_1) \qquad 0 \leqq k \leqq 5 \qquad (s_5 = 0)$$

$$s_o = af_2(y_1) + a^2f_3(y_1) + a^3f_4(y_1) + a^4f_5(y_1) + a^5f_6(y_1) + \cdots$$

$$s_1 = af_2'(y_1) + a^2f_3'(y_1) + a^3f_4'(y_1) + a^4f_5'(y_1) + \ldots$$

$$- \left[2af_3(y_1) + 3a^2f_4(y_1) + 4a^3f_5(y_1) + 5a^4f_6(y_1) + \ldots\right]$$

$$s_2 = \frac{1}{2}\left[af_2''(y_1) + a^2f_3''(y_1) + a^3f_4''(y_1) + \ldots\right]$$

$$- \left[2af_3'(y_1) + 3a^2f_4'(y_1) + 4a^3f_5'(y_1) + \ldots\right]$$

$$+ \left[3af_4(y_1) + 6a^2f_5(y_1) + 10a^3f_6(y_1) + \ldots\right]$$

$$s_3 = \frac{1}{6}\left[af_2'''(y_1) + a^2f_3'''(y_1) + \ldots\right] - \frac{1}{2}\left[2af_3''(y_1) + 3a^2f_4''(y_1) + \ldots\right]$$

$$+ \left[3af_4'(y_1) + 6a^2f_5'(y_1) + \ldots\right] - \left[4af_5(y_1) + 10a^2f_6(y_1) + \ldots\right]$$

$$s_4 = \frac{1}{24}\left[af_2^{iv}(y_1) + \ldots\right] - \frac{1}{6}\left[2af_3'''(y_1) + \ldots\right] + \frac{1}{2}\left[3af_4''(y_1) + \ldots\right]$$

$$- \left[4af_5'(y_1) + \ldots\right] + \left[5af_6(y_1) + \ldots\right]$$

$$d_k = -\sum_{r=1}^{k} \frac{A_{r+1}}{(r + 1)!} d_{k-r} \qquad d_o = 1.0$$

$$A_k = \frac{w_1^{(k)}}{w_1'} \qquad a = 0.40$$

(b) Define

$$K_{12}(c) = \int_{y_j}^{y_2} \frac{T}{(w - c)^2} dy$$

$$= \int_{0.40}^{1.0} \frac{T}{(w - c)^2} d(y - y_1)$$

$$= \sum_{k=0}^{\infty} a_k(k + 1)c^k$$

where

$$a_k = \int_{0.40}^{1.0} \frac{T}{w^{k+2}} d(y - y_1)$$

The velocity profile $w(y)$ is approximated by a parabolic arc in the interval $0.40 \leqq y - y_1 \leqq y_3 - y_1$ and by a straight line $\left(w = \text{Constant} = w(y_3)\right)$ in the interval $y_3 - y_1 \leqq y - y_1 \leqq 1.0$. The value of y_3 is determined by imposing the condition that the area under the parabolic-arc straight-line segment equals the area under the actual velocity profile $w(y)$ in the interval $0.40 \leqq y - y_1 \leqq 1.0$. The parabolic arc $w = l + m(y - y_1) + n(y - y_1)^2$ is determined by the following conditions:

when $y = y_4 < 1$,

$$w = 1$$

$$w' = 0$$

when $y = y_j$ and $y_j - y_1 = 0.40$,

$$w = w\left(y_j\right)$$

where $w\left(y_j\right)$ is read off the velocity profile of figures 1 and 2. The value of y_4 is chosen so that the parabolic arc fits the velocity curve $w(y)$ closely over the widest possible range.

For $\sigma = 1$,

$$T = T_1 - \left[\left(T_1 - 1\right) - \frac{\gamma - 1}{2} M_o^2\right] w - \frac{\gamma - 1}{2} M_o^2 w^2$$

Therefore

$$a_k = T_1\left(I_{k+2} + J_{k+2}\right) - \left[\left(T_1 - 1\right) - \frac{\gamma - 1}{2} M_o^2\right]\left(I_{k+1} + J_{k+1}\right) - \frac{\gamma - 1}{2} M_o^2\left(I_k + J_k\right)$$

where

$$I_k = \int_{0.40}^{y_3 - y_1} \frac{d\left(y - y_1\right)}{w^k}$$

and

$$J_k = \int_{y_3 - y_1}^{1.0} \frac{d\left(y - y_1\right)}{\left[w\left(y_3\right)\right]^k} = \frac{1 - \left(y_3 - y_1\right)}{\left[w\left(y_3\right)\right]^k}$$

I_k is evaluated by approximating $w(y)$ by a parabolic arc as follows:

$$I_1 = \frac{1}{\sqrt{A}} \ln \left[\frac{\sqrt{A} - m - 2n(y - y_1)}{\sqrt{A} + m + 2n(y - y_1)}\right]_{0.40}^{y_3 - y_1}$$

$$I_k = -\frac{1}{(k-1)A}\left[\frac{m + 2n(y - y_1)}{\left[l + m(y - y_1) + n(y - y_1)^2\right]^{k-1}}\right]_{0.40}^{y_3 - y_1} + \frac{2k-3}{2k-2}\,\frac{4(-n)}{A}\,I_{k-1}$$

where $A = m^2 - 4ln$.

As a control in the calculation of the series expression $\sum_{k=0}^{\infty} a_k(k+1)c^k$ for $K_{12}(c)$, use is made of the fact that, from the definition of I_k and J_k,

$$\lim_{k \to \infty}\left(I_k + J_k\right) = \frac{1}{k\left[\frac{w'(y_j)}{w(y_j)}\right]\left[w(y_j)\right]^k}$$

and therefore

$$\lim_{k \to \infty}\left(\frac{a_{k+1}}{a_k}\right) = \frac{1}{w(y_j)}\,\frac{k}{k+1}$$

The remainder after N - terms in the series for $K_{12}(c)$ is given approximately by

$$\frac{\left[(N+1)\text{ term}\right]}{\left[1-\frac{c}{w(y_j)}\right]}$$

The real part of $K_1(c)$ is obtained by combining the results of (a) and (b); that is,

$$\text{R.P. } K_1(c) = \text{R.P.}\left[K_{11}(c) + \frac{T_1}{w_1'c}\right] + K_{12}(c) - M_o^2$$

(2) Evaluation of $H_1(c)$:

$$H_1(c) = \int_{y_1}^{y_2} \frac{(w-c)^2}{T}\, dy$$

The integrand of this integral is free of singularities in the region of the complex y-plane bounded by $y = y_1$ and $y = y_2$; therefore $H_1(c)$ is evaluated by purely numerical integration. The actual procedure employed for the calculation of integrals of this type is as follows: (The integral $H_1(c)$ serves as an illustration.)

(a) Define

$$H_1(c) = \frac{1}{b}\left(\int_0^b \rho w^2\, d\eta - 2c\int_0^b \rho w\, d\eta + c^2\int_0^b \rho\, d\eta\right)$$

where

$$\eta = y^*\sqrt{\frac{\overline{u_o^*}}{\overline{\upsilon_o^*}\, x^*}}$$

and

$$b = \delta \sqrt{\frac{\overline{u_o}^*}{\overline{\upsilon_o}^* \; x^*}}$$

(b) With the approximation that the viscosity varies linearly with the absolute temperature, the velocity w is the same function of the nondimensional stream function ζ as in the Blasius flow; that is,

$$w = w(\zeta) = w_B(\zeta)$$

where ζ is defined by the relation $d\zeta = \rho w \, d\eta$ (appendix B).

From these relations

$$\rho w^n \, d\eta = \left[w(\zeta)\right]^{n-1} d\zeta = \left[w_B(\eta_B)\right]^n d\eta_B$$

since $d\zeta = w_B \, d\eta_B$. Moreover,

$$d\eta = \frac{d\zeta}{\rho w(\zeta)} = T(w_B) \; d\eta_B$$

where

$$T(w_B) = T_1 - \left[(T_1 - 1) - \frac{\gamma - 1}{2} M_o^2\right] w_B - \frac{\gamma - 1}{2} M_o^2 w_B^2$$

for $\sigma = 1$.

(c) Finally, from the relations given in (b),

$$H_1(c) = \frac{1}{b}\left(\int_0^{b_o} w_B{}^2 \, d\eta_B - 2c \int_0^{b_o} w_B \, d\eta_B + c^2\right)$$

where b_o is the value of $\delta\sqrt{\frac{\overline{u_o{}^*}}{\overline{\upsilon_o{}^*}\; x^*}}$ for the Blasius flow. For the insulated surfaces, b_o, which is somewhat arbitrary, was chosen as 5.60; whereas for the noninsulated surfaces, $b_o = 6.00$. (The value of w_B at $\eta_B = 5.60$ is 0.9950; when $\eta_B = 6.00$, $w_B = 0.9975$. The value of b for the insulated surfaces is the value of η at which $w = 0.9950$; whereas b for the noninsulated surfaces is the value of η for which $w = 0.9975$.) The advantage of this procedure is that the integrals $\int_0^{b_o} w_B{}^n \, d\eta_B$ are calculated once and for all and the value of $H_1(c)$ depends only upon the values of b and c. In fact,

$$\int_0^{b_o} w_B{}^2 \, d\eta_B = b_o - \left(\delta^*\sqrt{\frac{\overline{u_o{}^*}}{\overline{\upsilon_o{}^*}\; x^*}}\right)_B - \left(\theta\sqrt{\frac{\overline{u_o{}^*}}{\overline{\upsilon_o{}^*}\; x^*}}\right)_B = b_o - 2.3967$$

since

$$\left(\delta^*\sqrt{\frac{\overline{u_o{}^*}}{\overline{\upsilon_o{}^*}\; x^*}}\right)_B = 1.730$$

and

$$\left(\theta\sqrt{\frac{\overline{u_o{}^*}}{\overline{\upsilon_o{}^*}\; x^*}}\right)_B = 0.6667$$

Also,

$$\int_0^{b_o} w_B \, d\eta_B = b_o - 1.730$$

and

$$b = \int_0^b d\eta = \int_0^{b_o} T \, d\eta_B$$

$$= b_o + 1.73\left(T_1 - 1\right) + 0.6667 \frac{\gamma - 1}{2} M_o^2$$

$$= b_o + 1.73 \left[\left(T_1 - 1\right) - \frac{\gamma - 1}{2} M_o^2\right] + 2.3967 \frac{\gamma - 1}{2} M_o^2$$

See appendix B. (Incidentally, the last relation shows the effect of free-stream Mach number and thermal conditions at the solid surface on the "thickness" of the boundary layer.)

(3) Evaluation of $H_2(c)$:

$$H_2(c) = \int_{y_1}^{y_2} \frac{T - M_o^2(w - c)^2}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} dy$$

$$= \int_{y_1}^{y_2} \frac{T}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} dy - M_o^2 \int_{y_1}^{y_2} \int_{y_1}^{y} \frac{(w - c)^2}{T} dy \, dy$$

Define

$$H_{2_1}(c) = \int_{y_1}^{y_2} \frac{T}{(w - c)^2}\, dy \int_{y_1}^{y} \frac{(w - c)^2}{T}\, dy$$

$$H_{2_2}(c) = \int_{y_1}^{y_2} \int_{y_1}^{y} \frac{(w - c)^2}{T}\, dy\, dy$$

(a) The integral $H_{2_2}(c)$ is evaluated by methods similar to those already outlined for the evaluation of $H_1(c)$. Thus

$$H_{2_2}(c) = \int_{y_1}^{y_2} \int_{y_1}^{y} \frac{(w - c)^2}{T}\, dy\, dy$$

$$= \int_{y_1}^{y_2} dy \int_{y_1}^{y} \rho w^2\, dy - 2c \int_{y_1}^{y_2} dy \int_{y_1}^{y} \rho w\, dy + c^2 \int_{y_1}^{y_2} dy \int_{y_1}^{y} \rho\, dy$$

$$= \frac{1}{b^2}\left(\int_0^{b_o} T\, d\eta_B \int_0^{\eta_B} w_B{}^2\, d\eta_B - 2c \int_0^{b_o} T\, d\eta_B \int_0^{\eta_B} w_B\, d\eta_B + c^2 \int_0^{b_o} T\eta_B\, d\eta_B\right)$$

where

$$T = T_1 - \left[\left(T_1 - 1\right) - \frac{\gamma - 1}{2} M_o{}^2\right] w_B - \frac{\gamma - 1}{2} M_o{}^2 w_B{}^2$$

The nine integrals in the expression for $H_{2_2}(c)$ are evaluated by numerical integration using Simpson's rule.

(b) Define

$$H_{2_1}(c) = \int_{y_1}^{y_2} \frac{T}{(w - c)^2}\,dy \int_{y_1}^{y} \frac{(w - c)^2}{T}\,dy$$

$$= \int_{y_1}^{y_j} \frac{T}{(w-c)^2}\,dy \int_{y_1}^{y} \frac{(w-c)^2}{T}\,dy + \int_{y_j}^{y_2} \frac{T}{(w-c)^2}\,dy \int_{y_1}^{y} \frac{(w-c)^2}{T}\,dy$$

Define

$$H_{2_{11}}(c) = \int_{y_1}^{y_j} \frac{T}{(w - c)^2}\,dy \int_{y_1}^{y} \frac{(w - c)^2}{T}\,dy$$

$$H_{2_{12}}(c) = \int_{y_j}^{y_2} \frac{T}{(w - c)^2}\,dy \int_{y_1}^{y} \frac{(w - c)^2}{T}\,dy$$

The integral $H_{2_{12}}(c)$ is evaluated as follows:

$$H_{2_{12}}(c) = \int_{y_j}^{y_2} \frac{T}{(w - c)^2}\,dy \int_{y_1}^{y} \frac{(w - c)^2}{T}\,dy$$

$$= \int_{y_j}^{y_2} \frac{T}{(w-c)^2}\,dy \int_{y_1}^{y_2} \frac{(w-c)^2}{T}\,dy - \int_{y_j}^{y_2} \frac{T}{(w-c)^2}\,dy \int_{y}^{y_2} \frac{(w-c)^2}{T}\,dy$$

But

$$\int_{y_j}^{y_2} \frac{T}{(w - c)^2}\, dy = K_{12}(c)$$

and

$$\int_{y_1}^{y_2} \frac{(w - c)^2}{T}\, dy = H_1(c)$$

so that

$$H_{2_{12}}(c) = K_{12}(c)\, H_1(c) - \int_{y_j}^{y_2} \frac{T}{(w - c)^2}\, dy \int_{y}^{y_2} \frac{(w - c)^2}{T}\, dy$$

Define

$$P(c) = \int_{y_j}^{y_2} \frac{T}{(w - c)^2}\, G(y;\ c)\, dy$$

$$= \frac{1}{b^2} \int_{0.4b}^{b} \frac{T}{(w - c)^2}\, G(\eta;\ c) d\eta$$

where

$$G(\eta;\ c) = \int_{\eta}^{b} \frac{(w - c)^2}{T}\,d\eta$$

$$= \int_{\eta}^{b} \frac{w^2}{T}\,d\eta - 2c\int_{\eta}^{b} \frac{w}{T}\,d\eta + c^2\int_{\eta}^{b} \frac{d\eta}{T}$$

and

$$T = T_1 - \left[\left(T_1 - 1\right) - \frac{\gamma - 1}{2} M_o^2\right] w - \frac{\gamma - 1}{2} M_o^2 w^2$$

The integral $P(c)$ is evaluated by numerical integration using Simpson's rule; the required values of w are read directly off the velocity profiles of figures 1 and 2. Finally,

$$H_{2_{12}}(c) = K_{12}(c)\ H_1(c) - P(c)$$

The integral $H_{2_{11}}(c)$ is evaluated in exactly the same way as $K_{11}(c)$ where

$$\frac{(w - c)^2}{T} = \left(w_c'\right)^2 \left(y - y_c\right)^2 \left(\frac{\psi^2}{T}\right)$$

and

$$\psi(y) = 1 + \frac{w_c''}{2w_c'}\left(y - y_c\right) + \frac{w_c'''}{3!\,w_c'}\left(y - y_c\right)^2 + \ldots$$

$$\text{R.P.}\left[H_{2_{11}}(c)\right] = \left(b_o a^2 + c_o a^3 + d_o a^4 + n_o a^5\right)$$

$$+ \sigma\left(b_1 a^2 + c_1 a^3 + d_1 a^4 - 2ab_o - 3a^2 c_o - 4a^3 d_o - 5a^4 n_o\right)$$

$$+ \sigma^2\left(b_2 a^2 + c_2 a^3 - 2ab_1 - 3a^2 c_1 - 4a^3 d_1 - \frac{1}{3} + 3ac_o + 6a^2 d_o + 10a^3 n_o\right)$$

$$+ \sigma^3\left(b_3 a^2 - 2ab_2 - 3a^2 c_2 + 3ac_1 + 6a^2 d_1 + p_o - 4ad_o - 10a^2 n_o + a_3\right)$$

$$+ \sigma^4\left(-2ab_3 + 3ac_2 + p_1 - 4ad_1 + q_o + 5an_o + a_4\right)$$

$$+ \sigma^5\left[p_2 + q_1 - \frac{h_3 f_o(w_c')^2}{36}\right] - \frac{1}{a-\sigma}\left\{\frac{\sigma^3}{3} - \sigma^4\left[\frac{1}{3}\frac{f_o'(y_1)}{f_o(y_1)} + \frac{1}{4}\left(A_2 - \frac{T_1'}{T_1}\right)\right]\right\}$$

$$+ \ln\left(\frac{a-\sigma}{\sigma}\right)\left(\frac{1}{3}\frac{f_1(y_1)}{f_o(y_1)}\sigma^3 - \sigma^4\left\{\frac{1}{3}\frac{f_1(y_1)f_o'(y_1)}{\left[f_o(y_1)\right]^2} + \frac{1}{4}\frac{f_1(y_1)}{f_o(y_1)}\left(A_2 - \frac{T_1'}{T_1}\right) - \frac{1}{3}\frac{f_1'(y_1)}{f_o(y_1)}\right\}\right)$$

where

$$b_o = \frac{(w_1')^2}{6}$$

$$b_1 = \frac{A_2}{3}(w_1')^2$$

$$b_2 = \left(\frac{A_3 + B_2}{6}\right)(w_1')^2$$

$$b_3 = \left(\frac{A_4}{18} + \frac{A_2A_3}{6}\right)\left(w_1'\right)^2 - \frac{f_3(y_1)}{3f_o(y_1)}$$

$$a_3 = \frac{f_2(y_1)}{3f_o(y_1)}$$

$$a_4 = -\left\{\frac{1}{3}\frac{f_2(y_1)f_o'(y_1)}{\left[f_o(y_1)\right]^2} + \frac{1}{4}\frac{f_2(y_1)}{f_o(y_1)}\left(A_2 - \frac{T_1'}{T_1}\right) - \frac{1}{3}\frac{f_2'(y_1)}{f_o(y_1)}\right\}$$

$$c_o = -\frac{1}{9}\frac{f_1(y_1)}{f_o(y_1)} - \frac{1}{12}\left(A_2 - \frac{T_1'}{T_1}\right)$$

$$c_1 = -\left\{\frac{1}{9}\frac{f_1'(y_1)}{f_o(y_1)} - \frac{1}{9}\frac{f_1(y_1)f_o'(y_1)}{\left[f_o(y_1)\right]^2} + \frac{1}{12}\left[A_2' - \frac{T_1''}{T_1} + \left(\frac{T_1'}{T_1}\right)^2\right]\right\}$$

$$c_2 = -\left\{\frac{1}{9}\frac{f_1(y_1)}{f_o(y_1)}\left[\frac{f_o'(y_1)}{f_o(y_1)}\right]^2 - \frac{1}{9}\frac{f_1'(y_1)}{f_o(y_1)}\frac{f_o'(y_1)}{f_o(y_1)} + \frac{1}{18}\frac{f_1''(y_1)}{f_o(y_1)}\right.$$

$$\left. - \frac{1}{18}\frac{f_1(y_1)}{f_o(y_1)}\frac{f_o''(y_1)}{f_o(y_1)} + \frac{1}{24}\left[A_2'' - \frac{T_1'''}{T_1} + 3\frac{T_1''}{T_1}\frac{T_1'}{T_1} - 2\left(\frac{T_1'}{T_1}\right)^3\right]\right\}$$

$$d_o = -\frac{1}{12}\frac{f_2(y_1)}{f_o(y_1)} - \frac{1}{16}\frac{f_1(y_1)}{f_o(y_1)}\left(A_2 - \frac{T_1'}{T_1}\right) + \frac{1}{40}\left[\frac{B_2}{2} + \frac{2}{3}A_3 - 2A_2\frac{T_1'}{T_1} - \frac{T_1''}{T_1} + 2\left(\frac{T_1'}{T_1}\right)^2\right]$$

$$d_1 = -\frac{1}{12}\frac{f_2'(y_1)}{f_o(y_1)} + \frac{1}{12}\frac{f_2(y_1)}{f_o(y_1)}\frac{f_o'(y_1)}{f_o(y_1)} + \frac{1}{40}\left[\frac{B_2'}{2} + \frac{2}{3}A_3' - 2A_2'\frac{T_1'}{T_1}\right.$$

$$- 2A_2\frac{T_1''}{T_1} + 2A_2\left(\frac{T_1'}{T_1}\right)^2 - \frac{T_1'''}{T_1} + 5\frac{T_1''}{T_1}\frac{T_1'}{T_1} - 4\left.\left(\frac{T_1'}{T_1}\right)^3\right]$$

$$-\frac{1}{16}\frac{f_1(y_1)}{f_o(y_1)}\left[A_2' - \frac{T_1''}{T_1} + \left(\frac{T_1'}{T_1}\right)^2\right] - \frac{1}{16}\left\{\left(A_2 - \frac{T_1'}{T_1}\right)\left[\frac{f_1'(y_1)}{f_o(y_1)}\right.\right.$$

$$\left.\left.-\frac{f_1(y_1)}{f_o(y_1)}\frac{f_o'(y_1)}{f_o(y_1)}\right]\right\}$$

$$p_o = \frac{1}{4}\left(A_2 - \frac{T_1'}{T_1}\right)$$

$$p_1 = \frac{1}{4}\left[A_2' - \frac{T_1''}{T_1} + \left(\frac{T_1'}{T_1}\right)^2\right]$$

$$p_2 = \frac{1}{8}\left[A_2'' - \frac{T_1'''}{T_1} + 3\frac{T_1''}{T_1}\frac{T_1'}{T_1} - 2\left(\frac{T_1'}{T_1}\right)^3\right]$$

$$q_o = -\frac{1}{10}\left[\frac{B_2}{2} + \frac{2}{3}A_3 - 2A_2\frac{T_1'}{T_1} - \frac{T_1''}{T_1} + 2\left(\frac{T_1'}{T_1}\right)^2\right]$$

$$q_1 = -\frac{1}{10}\left[\frac{B_2'}{2} + \frac{2}{3}A_3' - 2A_2'\frac{T_1'}{T_1} - 2A_2\frac{T_1''}{T_1} + 2A_2\left(\frac{T_1'}{T_1}\right)^2 - \frac{T_1'''}{T_1}\right.$$

$$\left. + 5\frac{T_1''}{T_1}\frac{T_1'}{T_1} - 4\left(\frac{T_1'}{T_1}\right)^3\right]$$

$$n_o = -\frac{2}{15}\frac{f_3(y_1)}{f_o(y_1)} - \frac{1}{20}\frac{f_2(y_1)}{f_o(y_1)}\left(A_2 - \frac{T_1'}{T_1}\right)$$

$$-\frac{1}{50}\frac{f_1(y_1)}{f_o(y_1)}\left[\frac{B_2}{2} + \frac{2}{3}A_3 - 2A_2\frac{T_1'}{T_1} - \frac{T_1''}{T_1} + 2\left(\frac{T_1'}{T_1}\right)^2\right]$$

$$+\frac{1}{180}\left\{B_3 - \frac{T_1'}{T_1}\left(\frac{3}{2}B_2 + 2A_3\right) + \frac{A_4}{2} + 3A_2\left[2\left(\frac{T_1'}{T_1}\right)^2 - \frac{T_1''}{T_1}\right]\right.$$

$$\left. - \frac{T_1'''}{T_1} + 6\left(\frac{T_1''}{T_1}\right)\left(\frac{T_1'}{T_1}\right) - 6\left(\frac{T_1'}{T_1}\right)^3\right\}$$

$$\frac{h_3 f_o\left(w_c'\right)^2}{36} = \frac{1}{36}\left\{B_3 - \left(\frac{T_1'}{T_1}\right)\left(\frac{3}{2}B_2 + 2A_3\right) + \frac{A_4}{2} + 3A_2\left[2\left(\frac{T_1'}{T_1}\right)^2 - \frac{T_1''}{T_1}\right]\right.$$

$$\left. - \frac{T_1'''}{T_1} + 6\left(\frac{T_1''}{T_1}\right)\left(\frac{T_1'}{T_1}\right) - 6\left(\frac{T_1'}{T_1}\right)^3\right\}$$

$$A_k = \frac{w_1^{(k)}}{w_1'}$$

$$A_k' = A_{k+1} - A_2A_k \qquad 2 \leqq k \leqq 6$$

$$A_k'' = A_{k+1}' - A_2A_k' - A_2'A_k \qquad 2 \leqq k \leqq 5$$

$$A_k''' = A_{k+1}'' - A_2A_k'' - 2A_2'A_k' - A_2''A_k \qquad 2 \leqq k \leqq 4$$

$$A_k^{iv} = A_{k+1}''' - A_2A_k''' - 3A_2'A_k'' - 3A_2''A_k' - A_2'''A_k \qquad k = 2,\ 3$$

$$A_k^{v} = A_{k+1}^{iv} - A_2A_k^{iv} - 4A_2'A_k''' - 6A_2''A_k'' - 4A_2'''A_k' - A_2^{iv}A_k \qquad k = 2$$

$$B_k^{(m)} = A_{k+1}^{(m)} - A_k^{(m+1)} \qquad 2 \leqq k \leqq 6,\quad 0 \leqq m \leqq 4$$

$$B_7 = A_3^2$$

$$B_7' = 2A_3A_3'$$

$$B_7'' = 2(A_3')^2 + 2A_3A_3''$$

$$B_8 = A_3A_4$$

$$B_8' = A_3'A_4 + A_3A_4'$$

$$B_9 = A_4^{\;2}$$

$$B_{10} = A_3A_5$$

Finally,

$$\text{R.P. } H_2(c) = \text{R.P. } H_{2_{11}}(c) + H_{2_{12}}(c) - M_o^{\;2}H_{22}(c)$$

(4) Evaluation of $M_3(c)$:

$$M_3(c) = \int_{y_1}^{y_2} \frac{(w - c)^2}{T}\,dy \int_{y}^{y_2}\left(\frac{T}{(w - c)^2} - M_o^{\;2}\right) dy \int_{y_1}^{y} \frac{(w - c)^2}{T}\,dy$$

$$M_3(c) = M_{31}(c) - M_o^{\;2}M_{32}(c)$$

where

$$M_{31}(c) = \int_{y_1}^{y_2} \frac{(w - c)^2}{T}\,dy \int_{y}^{y_2} \frac{T}{(w - c)^2}\,dy \int_{y_1}^{y} \frac{(w - c)^2}{T}\,dy$$

and

$$M_{32}(c) = \int_{y_1}^{y_2} \frac{(w - c)^2}{T}\,dy \int_{y}^{y_2} \int_{y_1}^{y} \frac{(w - c)^2}{T}\,dy\,dy$$

(a) The integral $M_{32}(c)$ is evaluated in much the same way as $H_{22}(c)$; that is,

$$b^3M_{32}(c) = \int_0^{b_o} w_B^{\,2}\,d\eta_B \int_{\eta_B}^{b_o}\int_0^{\eta_B} Tw_B^{\,2}\,d\eta_B\,d\eta_B$$

$$- 2c\left(\int_0^{b_o} w_B^{\,2}\,d\eta_B \int_{\eta_b}^{b_o}\int_0^{\eta_B} Tw_B\,d\eta_B\,d\eta_B + \int_0^{b_o} w_B\,d\eta_B \int_{\eta_B}^{b_o}\int_0^{\eta_B} Tw_B^{\,2}\,d\eta_B\,d\eta_B\right)$$

$$+ c^2\left(\int_0^{b_o} d\eta_B \int_{\eta_B}^{b_o}\int_0^{\eta_B} Tw_B^{\,2}\,d\eta_B\,d\eta_B + \int_0^{b_o} w_B^{\,2}\,d\eta_B \int_{\eta_B}^{b_o} T\eta_B\,d\eta_B\right.$$

$$\left. + 4\int_0^{b_o} w_B\,d\eta_B \int_{\eta_B}^{b_o}\int_0^{\eta_B} Tw_B\,d\eta_B\,d\eta_B\right) - 2c^3\left(\int_0^{b_o} w_B\,d\eta_B \int_{\eta_B}^{b_o} T\eta_B\,d\eta_B\right.$$

$$\left. + \int_0^{b_o} d\eta_B \int_{\eta_B}^{b_o}\int_0^{\eta_B} Tw_B\,d\eta_B\,d\eta_B\right) + c^4\int_0^{b_o} d\eta_B \int_{\eta_B}^{b_o} T\eta_B\,d\eta_B$$

where b_o has the same meaning as in the evaluation of $H_{22}(c)$ and where

$$T = T\left(w_B\right) = T_1 - \left[\left(T_1 - 1\right) - \frac{\gamma - 1}{2} M_o^{\,2}\right] w_B - \frac{\gamma - 1}{2} M_o^{\,2} w_B^{\,2}$$

The integrals in $M_{32}(c)$ are evaluated by numerical integration using Simpson's rule. Values of w_B are taken from the table in appendix B.

(b) For convenience, the integral $M_{31}(c)$ is transformed as follows:

$$M_{31}(c) = M_{31_1}(c) + M_{31_2}(c) - M_{31_3}(c)$$

where

$$M_{31_1}(c) = \int_{y_1}^{y_j} \frac{(w - c)^2}{T} dy \int_{y}^{y_j} \frac{T}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} dy$$

$$M_{31_2}(c) = \int_{y_1}^{y_2} \frac{(w - c)^2}{T} dy \int_{y_j}^{y_2} \frac{T}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} dy$$

$$M_{31_3}(c) = \int_{y_j}^{y_2} \frac{(w - c)^2}{T} dy \int_{y_j}^{y} \frac{T}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} dy$$

It is recognized that

$$\int_{y_1}^{y_2} \frac{(w - c)^2}{T} dy = H_1(c)$$

$$\int_{y_j}^{y_2} \frac{T}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} dy = H_{2_{12}}(c)$$

Therefore

$$M_{31_2}(c) = H_1(c)\ H_{2_{12}}(c)$$

By additional transformations, the following equations are obtained:

$$M_{31_3}(c) = H_1(c)\ P(c) - Q(c)$$

where

$$Q(c) = \int_{y_j}^{y_2} \frac{(w - c)^2}{T}\, dy \int_{y_j}^{y} \frac{T}{(w - c)^2}\, dy \int_{y}^{y_2} \frac{(w - c)^2}{T}\, dy$$

or

$$Q(c) = \frac{1}{b^3} \int_{0.4b}^{b} \frac{(w - c)^2}{T}\, d\eta \int_{0.4b}^{\eta} \frac{T}{(w - c)^2}\, d\eta \int_{\eta}^{b} \frac{(w - c)^2}{T}\, d\eta$$

The integral $Q(c)$ is evaluated by numerical integration using Simpson's rule; $P(c)$ is evaluated in the calculation of $H_{22}(c)$.

The integral $M_{31_1}(c)$ is obtained in exactly the same way as $K_{11}(c)$ and $H_{2_{11}}(c)$; that is,

$$\text{R.P. } M_{31_1}(c) = \frac{1}{f_o(y_1)}\left(-\frac{a^5}{45} + \sigma\frac{a^4}{9} - 2\sigma^2\frac{a^3}{9} + \sigma^4\frac{a}{3}\right)$$

$$+ \frac{\sigma^3}{3f_o(y_1)}\left(\frac{a^3}{3} + \sigma a^2 + \sigma^2 a\right)\left[\frac{1}{(a - \sigma)} + \frac{f_1(y_1)}{f_o(y_1)} \ln (a - \sigma)\right]$$

Finally,

$$\text{R.P. } M_3(c) = \text{R.P. } M_{31_1}(c) + H_1(c)\left[H_{2_{12}}(c) - P(c)\right] + Q(c) - M_o^2 M_{32}(c)$$

(5) Evaluation of $N_3(c)$:

$$N_3(c) = \int_{y_1}^{y_2}\left[\frac{T}{(w - c)^2} - M_o^2\right] dy \int_{y_1}^{y}\frac{(w - c)^2}{T} dy \int_{y}^{y_2}\left[\frac{T}{(w - c)^2} - M_o^2\right] dy$$

$$N_3(c) = \int_{y_1}^{y_2}\frac{T}{(w - c)^2} dy \int_{y_1}^{y}\frac{(w - c)^2}{T} dy \int_{y}^{y_2}\frac{T}{(w - c)^2} dy$$

$$- M_o^2\int_{y_1}^{y_2}\frac{T}{(w - c)^2} dy \int_{y_1}^{y}\left(y_2 - y\right)\frac{(w - c)^2}{T} dy$$

$$- M_o^2\int_{y_1}^{y_2} dy \int_{y_1}^{y}\frac{(w - c)^2}{T} dy \int_{y}^{y_2}\frac{T}{(w - c)^2} dy$$

$$+ M_o^4\int_{y_1}^{y_2} dy \int_{y_1}^{y}\frac{(w - c)^2}{T}\left(y_2 - y\right) dy$$

It can be shown that the second and third integrals are identical; therefore

$$N_3(c) = N_{31}(c) - 2M_o^2 N_{32}(c) + M_o^4 N_{34}(c)$$

where

$$N_{31}(c) = \int_{y_1}^{y_2} \frac{T}{(w - c)^2}\, dy \int_{y_1}^{y} \frac{(w - c)^2}{T}\, dy \int_{y}^{y_2} \frac{T}{(w - c)^2}\, dy$$

$$N_{32}(c) \equiv N_{33}(c) = \int_{y_1}^{y_2} \frac{T}{(w - c)^2}\, dy \int_{y_1}^{y} \left(y_2 - y\right) \frac{(w - c)^2}{T}\, dy$$

$$N_{34}(c) = \int_{y_1}^{y_2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} \left(y_2 - y\right) dy$$

(a) The integral $N_{34}(c)$ is evaluated by numerical integration in a manner similar to $H_1(c)$, $H_{22}(c)$, and $M_{32}(c)$; that is,

$$N_{34}(c) = \int_{y_1}^{y_2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T}\, dy \int_{y}^{y_2} dy$$

$$= \frac{1}{b^3}\left(\int_0^b d\eta \int_0^\eta \rho w^2\, d\eta \int_\eta^b d\eta - 2c \int_0^b d\eta \int_0^\eta \rho w\, d\eta \int_\eta^b d\eta + c^2 \int_0^b d\eta \int_0^\eta \rho d\eta \int_\eta^b d\eta\right)$$

$$N_{34}(c) = \frac{1}{b^3}\left(\int_0^{b_o} T\, d\eta_B \int_0^{\eta_B} w_B^2\, d\eta_B \int_{\eta_B}^{b_o} T\, d\eta_B - 2c\int_0^{b_o} T\, d\eta_B \int_0^{\eta_B} w_B\, d\eta_B \int_{\eta_B}^{b_o} T\, d\eta_B\right.$$

$$\left. + c^2 \int_0^{b_o} T\, d\eta_B \int_0^{\eta_B} d\eta_B \int_{\eta_B}^{b_o} T\, d\eta_B\right)$$

where

$$T = T_1 - \left[\left(T_1 - 1\right) - \frac{\gamma - 1}{2} M_o^2\right] w_B - \frac{\gamma - 1}{2} M_o^2 w_B^2$$

The integrals in $N_{34}(c)$ are evaluated by numerical integration in a manner similar to that used in the evaluation of $M_{32}(c)$, and so forth. Most of the integrals will already have been evaluated in the calculation of $H_1(c)$, $H_{22}(c)$, and $M_{32}(c)$.

(b) For convenience, $N_{32}(c)$ is broken down as follows:

$$N_{32}(c) = \int_{y_1}^{y_2} \frac{T}{(w - c)^2}\, dy \int_{y_1}^{y} \frac{(w - c)^2}{T} \left(y_2 - y\right) dy$$

$$= \int_{y_1}^{y_j} \frac{T}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} \left(y_2 - y\right) dy + \int_{y_j}^{y_2} \frac{T}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} \left(y_2 - y\right) dy$$

Let

$$N_{32}(c) = N_{32_1}(c) + N_{32_2}(c) - N_{32_3}(c) .$$

where

$$N_{32_1}(c) = \int_{y_1}^{y_j} \frac{T}{(w - c)^2}\, dy \int_{y_1}^{y} \frac{(w - c)^2}{T} \left(y_2 - y\right) dy$$

$$N_{32_2}(c) = \int_{y_j}^{y_2} \frac{T}{(w - c)^2}\, dy \int_{y_1}^{y_2} \frac{(w - c)^2}{T} \left(y_2 - y\right) dy$$

$$N_{32_3}(c) = \int_{y_j}^{y_2} \frac{T}{(w - c)^2}\, dy \int_{y}^{y_2} \frac{(w - c)^2}{T} \left(y_2 - y\right) dy$$

Now,

$$N_{32_3}(c) = \int_{y_j}^{y_2} \frac{T}{(w - c)^2}\, dy \int_{y}^{y_2} \frac{(w - c)^2}{T} \left[\left(y_2 - y_1\right) - \left(y - y_1\right)\right] dy$$

Since $y_2 - y_1 = 1.0$, and

$$P(c) = \int_{y_j}^{y_2} \frac{T}{(w - c)^2}\, dy \int_{y}^{y_2} \frac{(w - c)^2}{T}\, dy$$

it is found that

$$N_{32_3}(c) = P(c) - P_1(c)$$

where

$$P_1(c) = \int_{y_j}^{y_2} \frac{T}{(w - c)^2}\, dy \int_{y}^{y_2} \frac{(w - c)^2}{T} \left(y - y_1\right) dy$$

$$= \frac{1}{b^3} \int_{0.4b}^{b} \frac{T}{(w - c)^2}\, G_1(\eta;\ c)\, d\eta$$

and

$$G_1(\eta;\ c) = \int_{\eta}^{b} \frac{w^2}{T}\, \eta\, d\eta - 2c \int_{\eta}^{b} \frac{w}{T}\, \eta\, d\eta + c^2 \int_{\eta}^{b} \frac{\eta\, d\eta}{T}$$

$P_1(c)$ is evaluated by numerical integration using Simpson's rule. Define

$$N_{32_2}(c) = \int_{y_j}^{y_2} \frac{T}{(w - c)^2}\, dy \int_{y_1}^{y_2} \frac{(w - c)^2}{T} \left[\left(y_2 - y_1\right) - \left(y - y_1\right)\right] dy$$

Since

$$\int_{y_j}^{y_2} \frac{T}{(w - c)^2}\, dy = K_{12}(c)$$

and

$$\int_{y_1}^{y_2} \frac{(w - c)^2}{T}\, dy = H_1(c)$$

it is recognized that

$$N_{32_2}(c) = K_{12}(c)\left[H_1(c) - \int_{y_1}^{y_2} \frac{(w - c)^2}{T}\left(y - y_1\right) dy\right]$$

$$\int_{y_1}^{y_2}\frac{(w - c)^2}{T}\left(y - y_1\right)dy = \frac{1}{b^2}\int_0^b \rho(w - c)^2 d\eta \int_0^\eta d\eta = \frac{1}{b^2}\left[\int_0^b \rho\left(w^2 - 2cw + c^2\right)d\eta \int_0^\eta d\eta\right]$$

$$\int_{y_1}^{y_2}\frac{(w - c)^2}{T}\left(y - y_1\right)dy = \frac{1}{b^2}\left(\int_0^{b_o} w_B{}^2\, d\eta_B \int_0^{\eta_B} T\, d\eta_B - 2c\int_0^{\eta_B} w_B\, d\eta_B \int_0^{\eta_B} T\, d\eta_B\right.$$

$$\left. + c^2 \int_0^{\eta_B} d\eta_B \int_0^{\eta_B} T\, d\eta_B\right)$$

The integral $\int_{y_1}^{y_2} \frac{(w - c)^2}{T}\left(y - y_1\right)$ is evaluated by numerical integration in exactly the same way as $N_{34}(c)$.

The integral $N_{32_1}(c)$ is transformed as follows:

$$N_{32_1}(c) = \int_{y_1}^{y_j} \frac{T}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} \left[\left(y_2 - y_c\right) - \left(y - y_c\right)\right] dy$$

But

$$\int_{y_1}^{y_j} \frac{T}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} dy = H_{2_{11}}(c)$$

and

$$y_2 - y_c = \left(y_2 - y_1\right) - \left(y_c - y_1\right) = 1 - \sigma$$

so that

$$N_{32_1}(c) = (1 - \sigma)H_{2_{11}}(c) - J_{2_{11}}(c)$$

where

$$J_{2_{11}}(c) = \int_{y_1}^{y_j} \frac{T}{(w - c)^2} dy \int_{y_1}^{y} \frac{(w - c)^2}{T} \left(y - y_c\right) dy$$

The integral $J_{2_{11}}(c)$ is evaluated in the same way as $K_{11}(c)$. Thus

$$\text{R.P. } J_{2_{11}}(c) = \frac{1}{a - \sigma}\left[\frac{\sigma^4}{4} + \sigma^5\left(-\frac{4}{5}\,p_o\right)\right]$$

$$+ \ln\left(\frac{a - \sigma}{\sigma}\right)\left[\frac{1}{4}\,\frac{f_1(y_1)}{f_o(y_1)}\sigma^4 + \sigma^5\left(-\frac{4}{5}\,\frac{f_1(y_1)}{f_o(y_1)}p_o + \frac{1}{4}\left\{\frac{f_1{}'(y_1)}{f_o(y_1)} - \frac{f_1(y_1)f_o{}'(y_1)}{\left[f_o(y_1)\right]^2}\right\}\right)\right]$$

$$+ \frac{1}{12}\,a^3 + a^4C_o - a^5D_o + \sigma\left[-\frac{a^2}{4} - 4a^3C_o + a^4\left(5D_o + C_1\right)\right]$$

$$+ \sigma^2\left(\frac{a}{4} + 6a^2C_o - 10a^3D_o - 4a^3C_1\right) + \sigma^3\left(\frac{1}{4} - 4aC_o + 10a^2D_o + 6a^2C_1\right)$$

$$+ \sigma^4\left[-\frac{4}{5}\,p_o + \frac{a}{4}\,\frac{f_2(y_1)}{f_o(y_1)} - 4aC_1 - 5aD_o\right] + \sigma^5\left(-\frac{5}{6}\,q_o\right)$$

where

$$C_o = \frac{p_o}{5} - \frac{1}{16}\,\frac{f_1(y_1)}{f_o(y_1)}$$

$$C_1 = \frac{p_1}{5} - \frac{1}{16}\left\{\frac{f_1{}'(y_1)}{f_o(y_1)} - \frac{f_1(y_1)f_o{}'(y_1)}{\left[f_o(y_1)\right]^2}\right\}$$

$$D_o = \frac{q_o}{6} + \frac{4}{25}\,\frac{f_1(y_1)}{f_o(y_1)}\,p_o + \frac{1}{20}\,\frac{f_2(y_1)}{f_o(y_1)}$$

Finally

$$\text{R.P. } N_{32}(c) = \text{R.P. } N_{32_1}(c) + N_{32_2}(c) - N_{32_3}(c)$$

(c) Define

$$N_{31}(c) = \int_{y_1}^{y_2} \frac{T}{(w - c)^2}\, dy \int_{y_1}^{y} \frac{(w - c)^2}{T}\, dy \int_{y}^{y_2} \frac{T}{(w - c)^2}\, dy$$

After several transformations, the integral $N_{31}(c)$ is brought into a more convenient form

$$N_{31}(c) = N_{31_1}(c) + K_{12}(c)\left[2H_{2_{11}}(c) + H_{2_{12}}(c)\right] - N_{31_3}(c)$$

where

$$N_{31_1}(c) = \int_{y_1}^{y_j} \frac{T}{(w - c)^2}\, dy \int_{y_1}^{y} \frac{(w - c)^2}{T}\, dy \int_{y}^{y_j} \frac{T}{(w - c)^2}\, dy$$

$$N_{31_3}(c) = \int_{y_j}^{y_2} \frac{T}{(w - c)^2}\, dy \int_{y_j}^{y} \frac{(w - c)^2}{T}\, dy \int_{y_j}^{y} \frac{T}{(w - c)^2}\, dy$$

The integral $N_{31_3}(c)$ is evaluated by numerical integration using Simpson's rule. Some of the integrations have already been performed. The integral is given as

$$N_{31_3}(c) = \frac{1}{b^3} \int_{0.4b}^{b} \frac{T}{(w - c)^2}\, d\eta \int_{0.4b}^{\eta} \frac{(w - c)^2}{T}\, d\eta \int_{0.4b}^{\eta} \frac{T}{(w - c)^2}\, d\eta$$

The integral $N_{31_1}(c)$ is evaluated in exactly the same way as $K_{11}(c)$; that is,

$$N_{31_1}(c) = \frac{1}{(a-\sigma)^2}\left\{-\frac{\sigma^3}{3}f_o(y_1) + \sigma^4\left[p_of_o(y_1) - \frac{1}{3}f_o'(y_1)\right] + \sigma^5\left[(p_1 - q_o)f_o(y_1)\right.\right.$$

$$\left.\left. + p_of_o'(y_1) - \frac{1}{6}f_o''(y_1)\right]\right\} - \left[\ln(a-\sigma)\right]^2 B - \frac{2\ln(a-\sigma)}{(a-\sigma)}A$$

$$+ \frac{1}{a-\sigma}\left(A\ln\sigma + \frac{a^2}{6}f_o(y_1) - a^3\left[\frac{1}{3}p_of_o(y_1) + \frac{1}{9}f_1(y_1)\right] + \frac{a^4}{4}F\right.$$

$$+ \sigma\left\{\frac{a^2}{6}f_o'(y_1) - \frac{a}{3}f_o(y_1) + a^2\left[p_of_o(y_1) + \frac{1}{3}f_1(y_1)\right] - \frac{a^3}{3}E - a^3F\right\}$$

$$+ \sigma^2\left\{-\frac{1}{3}f_o(y_1) + \frac{a^2}{12}f_o''(y_1) - a\left[\frac{1}{3}f_o'(y_1) + p_of_o(y_1) + \frac{1}{3}f_1(y_1)\right]\right.$$

$$\left. + a^2E + \frac{3a^2}{2}F\right\} + \sigma^3\left[p_of_o(y_1) - \frac{1}{3}f_o'(y_1) - \frac{a}{3}f_2(y_1) - \frac{a}{6}f_o''(y_1) - a(E+F)\right]$$

$$\left. + \sigma^4\left[(p_1 - q_o)f_o(y_1) + p_of_o'(y_1) - \frac{1}{6}f_o''(y_1)\right]\right)$$

$$+ \ln(a-\sigma)\left[B\ln\sigma + \frac{a^2}{6}f_1(y_1) - \frac{a^3}{3}\left\{p_of_1(y_1) + \frac{1}{3}\frac{\left[f_1(y_1)\right]^2}{f_o(y_1)}\right\} + \frac{a^4}{4}C\right.$$

$$+ \sigma\left(\frac{a^2}{6}f_1'(y_1) - \frac{a}{3}f_1(y_1) - \frac{a^3}{3}D + a^2\left\{p_of_1(y_1) + \frac{1}{3}\frac{\left[f_1(y_1)\right]^2}{f_o(y_1)}\right\} - a^3C\right)$$

$$+\sigma^2\left(-\frac{1}{3}f_1(y_1)+\frac{a^2}{12}f_1''(y_1)-\frac{a}{3}f_1'(y_1)+a^2D-a\left\{p_of_1(y_1)+\frac{1}{3}\frac{\left[f_1(y_1)\right]^2}{f_o(y_1)}\right\}\right.$$

$$\left.+\frac{3}{2}a^2C\right)+\sigma^3\left[p_of_1(y_1)-\frac{1}{3}f_1'(y_1)-\frac{a}{3}\frac{f_2(y_1)f_1(y_1)}{f_o(y_1)}-\frac{a}{6}f_1''(y_1)-a(D+C)\right]$$

$$\left.-\sigma^4\left[\frac{1}{6}f_1''(y_1)-p_of_1'(y_1)+(q_o-p_1)f_1(y_1)\right]\right]$$

$$-\frac{a}{2}f_o(y_1)-\frac{\sigma}{2}\left[f_o(y_1)+af_o'(y_1)\right]+\frac{\sigma^2}{2}f_1(y_1)\ln\sigma+\frac{(a-\sigma)^2}{6}\left[\frac{7}{3}f_1(y_1)\right.$$

$$\left.-4f_o(y_1)p_o\right]+(a-\sigma)^3\left\{\frac{5}{12}f_2(y_1)+\frac{17}{36}f_1(y_1)p_o-\frac{2}{27}\frac{\left[f_1(y_1)\right]^2}{f_o(y_1)}\right.$$

$$\left.+\frac{5}{12}f_o(y_1)q_o\right\}+\frac{(a-\sigma)^2\sigma}{6}\left[\frac{7}{3}f_1'(y_1)-4p_1f_o(y_1)-4p_of_o'(y_1)\right]$$

$$-\frac{\sigma^2}{6}(a-\sigma)f_2(y_1)-\frac{1}{6}(a-\sigma)^2\ln(a-\sigma)f_1(y_1)$$

$$-\frac{1}{6}(a-\sigma)^2\sigma f_1'(y_1)\ln(a-\sigma)+(a-\sigma)^3\ln(a-\sigma)\left\{\frac{1}{9}\frac{\left[f_1(y_1)\right]^2}{f_o(y_1)}\right.$$

$$\left.-\frac{1}{3}f_1(y_1)p_o\right\}+\frac{1}{a-\sigma}\left(-\frac{\sigma^2}{2}f_o(y_1)-\sigma^3\left[\frac{f_o'(y_1)}{2}-\frac{4}{3}f_o(y_1)p_o+\frac{1}{9}f_1(y_1)\right]\right.$$

$$- \sigma^4 \left\{\frac{1}{4} f_o''(y_1) + \frac{1}{9} f_1'(y_1) - \frac{4}{3}\left[p_o f_1'(y_1) + p_1 f_o(y_1)\right] + \frac{5}{4} q_o f_o(y_1) + \frac{1}{4} f_2(y_1)\right.$$

$$\left.\left. - \frac{f_1(y_1)p_o}{4}\right\}\right) + \ln(a - \sigma)\left[- \frac{\sigma^2}{2} f_1(y_1) - \sigma^3 \left\{\frac{f_1'(y_1)}{2} + \frac{1}{9} \frac{\left[f_1(y_1)\right]^2}{f_o(y_1)}\right.\right.$$

$$\left. - \frac{4}{3} f_1(y_1)p_o\right\} - \sigma^4 \left(\frac{1}{4} f_1''(y_1) + \frac{1}{9}\left\{\frac{2f_1(y_1)f_1'(y_1)}{f_o(y_1)} - \left[\frac{f_1(y_1)}{f_o\ y_1}\right]^2 f_o'(y_1)\right\}\right.$$

$$\left.\left. - \frac{4}{3}\left[f_1(y_1)p_1 + p_o f_1'(y_1)\right] + \frac{5}{4} f_1(y_1)q_o + \frac{1}{4} \frac{f_2(y_1)f_1(y_1)}{f_o(y_1)} - \frac{p_o}{4} \frac{\left[f_1(y_1)\right]^2}{f_o(y_1)}\right)\right]$$

where

$$A = \frac{\sigma^3}{3} f_1(y_1) + \sigma^4 \left[\frac{1}{3} f_1'(y_1) - p_o f_1(y_1)\right]$$

$$+ \sigma^5 \left[\frac{1}{6} f_1''(y_1) - p_o f_1'(y_1) + (q_o - p_1) f_1(y_1)\right]$$

$$B = \sigma^5 \left(\left(q_o - p_1\right)\left[\frac{f_1(y_1)}{f_o(y_1)}\right]^2 - p_o \left\{ \frac{2f_1(y_1)f_1'(y_1)}{f_o(y_1)} - \left[\frac{f_1(y_1)}{f_o(y_1)}\right]^2 f_o'(y_1) \right\} \right.$$

$$+ \frac{1}{3}\left\{\left[\frac{f_1'(y_1)}{f_o(y_1)}\right]^2 + \frac{f_1(y_1)f_1''(y_1)}{f_o(y_1)} - \frac{2f_1(y_1)f_1'(y_1)f_o'(y_1)}{\left[f_o(y_1)\right]^2}\right.$$

$$\left. \left. - \frac{1}{2}\left[\frac{f_1(y_1)}{f_o(y_1)}\right]^2 f_o''(y_1) + \left[\frac{f_1(y_1)}{f_o(y_1)}\right]^2 \frac{\left[f_o'(y_1)\right]^2}{f_o(y_1)} \right\} \right) + \frac{\sigma^3}{3} \frac{\left[f_1(y_1)\right]^2}{f_o(y_1)}$$

$$+ \sigma^4 \left\{ \frac{2}{3} \frac{f_1(y_1)f_1'(y_1)}{f_o(y_1)} - \frac{1}{3}\left[\frac{f_1(y_1)}{f_o(y_1)}\right]^2 f_o'(y_1) - p_o \frac{\left[f_1(y_1)\right]^2}{f_o(y_1)} \right\}$$

$$C = p_o \frac{\left[f_1(y_1)\right]^2}{f_o(y_1)} + q_o f_1(y_1) - \frac{1}{3} \frac{f_2(y_1)f_1(y_1)}{f_o(y_1)}$$

$$D = p_o f_1'(y_1) + p_1 f_1(y_1) + \frac{2}{3} \frac{f_1(y_1)f_1'(y_1)}{f_o(y_1)} - \frac{1}{3}\left[\frac{f_1(y_1)}{f_o(y_1)}\right]^2 f_o'(y_1)$$

$$E = p_o f_o'(y_1) + p_1 f_o(y_1) + \frac{1}{3} f_1'(y_1)$$

$$F = p_o f_1(y_1) + q_o f_o(y_1) - \frac{1}{3} f_2(y_1)$$

Evaluation of $f_k^{(m)}$

The functions $f_k^{(m)}(y_1)$, which appear repeatedly in the evaluation of the integrals $K_1(c)$, $H_2(c)$, and so forth, are evaluated in terms of $w_1^{(k)}$ and $T_1^{(k)}$ as follows:

$$f_k\, y_c = \frac{1}{\left(w_c'\right)^2 (k-1)k!}\left[\left(\frac{T}{\psi^2}\right)^k\right]_{y_c} = f_k\left(y_1\right) + f_k'\left(y_1\right)\sigma + \frac{f_k''\left(y_1\right)}{2!}\sigma^2 + \ldots$$

$$f_o^{(m)}\left(y_1\right) = -\left(Tg_o\right)_{y_1}^{(m)} = -\left[T_1^{(m)} g_o + mT_1^{(m-1)} g_1 + \frac{m(m-1)}{2!} T_1^{(m-2)} g_2 + \ldots\right.$$

$$\left. + \frac{m!}{(m-r)!r!} T_1^{(m-r)} g_r + \ldots + T_1 g_m\right] \qquad 0 \leqq m \leqq 6$$

where

$$g_o = \frac{1}{\left(w_1'\right)^2}$$

$$g_1 = -2g_o A_2$$

$$g_2 = -2\left(g_o A_2' + g_1 A_2\right)$$

$$\vdots$$

$$g_m = -2\left[g_o A_2^{(m-1)} + (m-1) g_1 A_2^{(m-2)}\right.$$

$$\left. + \ldots + \frac{m!}{(m-r)!r!} g_r A_2^{(m-r-1)} + \ldots + g_{m-1} A_2\right]$$

$$f_1^{(m)}(y_1) = \left(T'g_o + \frac{1}{2}\,g_1T\right)_{y_1}^{(m)} = \left(T'g_o\right)_{y_1}^{(m)} + \frac{1}{2}\left(g_1T\right)_{y_1}^{(m)} \qquad 0 \leqq m \leqq 5$$

$$f_2^{(m)}(y_1) = \frac{1}{2}\left(T'g_o\right)_{y_1}^{(m+1)} + \frac{1}{2}\left(Tg_oS_2\right)_{y_1}^{(m)} \qquad 0 \leqq m \leqq 4$$

where

$$\frac{1}{2}\left(Tg_oS_2\right)_{y_1}^{(m)} = -\frac{1}{2}\left[f_o(y_1)S_2\right]_{y_1}^{(m)}$$

and

$$S_2^{(k)} = \frac{3}{2}\,B_2^{(k)} - \frac{2}{3}\,A_3^{(k)} \qquad 0 \leqq k \leqq 4$$

$$f_3^{(m)}(y_1) = \frac{1}{12}\left(T'''g_o + \frac{3}{2}\,T''g_1\right)_{y_1}^{(m)} + \frac{1}{4}\left(T'g_oS_2\right)_{y_1}^{(m)}$$

$$-\frac{1}{12}\left[f_o(y_1)S_3\right]_{y_1}^{(m)} \qquad 0 \leqq m \leqq 3$$

$$f_4^{(m)}(y_1) = \frac{1}{72}\left(T^{iv}g_o + 2T'''g_1\right)_{y_1}^{(m)} + \frac{1}{12}\left(T''g_oS_2\right)_{y_1}^{(m)} + \frac{1}{18}\left(T'g_oS_3\right)_{y_1}^{(m)}$$

$$-\frac{1}{72}\left[f_o(y_1)S_4\right]_{y_1}^{(m)} \qquad 0 \leqq m \leqq 2$$

$$f_5^{(m)}(y_1) = \frac{1}{480}\left(T^{v}g_o + \frac{5}{2}\, g_1 T^{iv}\right)_{y_1}^{(m)} + \frac{1}{48}\left(T'''g_oS_2\right)_{y_1}^{(m)} + \frac{1}{48}\left(T''g_oS_3\right)_{y_1}^{(m)}$$

$$+ \frac{1}{96}\left(T'g_oS_4\right)_{y_1}^{(m)} - \frac{1}{480}\left[f_o(y_1)S_5\right]_{y_1}^{(m)} \qquad 0 \leqq m \leqq 1$$

$$f_6(y_1) = \frac{1}{3600}\left(T_1^{vi}g_o + 3T_1^{v}g_1\right) + g_o\left(\frac{1}{240}\, T_1^{iv}S_2 + \frac{1}{180}\, T_1'''S_3 + \frac{1}{240}\, T_1''S_4\right.$$

$$\left. + \frac{1}{600}\, T_1'S_5 - \frac{1}{3600}\, T_1S_6\right)$$

where

$$S_3^{(k)} = 3B_3^{(k)} - 3C_2^{(k)} - \frac{1}{2}A_4^{(k)} \qquad 0 \leqq k \leqq 3$$

$$S_4^{(k)} = -\frac{2}{5}A_5^{(k)} + 3B_4^{(k)} - 12C_3^{(k)} + \frac{15}{2}D_2^{(k)} + 2B_7^{(k)} \qquad 0 \leqq k \leqq 2$$

$$S_5^{(k)} = -\frac{1}{3}A_6^{(k)} + 3B_5^{(k)} + 5B_8^{(k)} - 15C_4^{(k)}$$

$$- 20C_6^{(k)} - 50D_3^{(k)} - \frac{45}{2}E_2^{(k)} \qquad 0 \leqq k \leqq 1$$

$$S_6 = -\frac{2}{7}A_7 + 3B_6 + 6B_{10} - 18C_5 + \frac{15}{4}B_9 - 60C_8 + 75D_4$$

$$-\frac{40}{3}C_7 + 150\ D_5 - 225\ E_3 + \frac{315}{4}F_2$$

$A_k^{(m)}$ and $B_k^{(m)}$ are defined as previously.

$$C_2^{(k)} = B_3^{(k)} - \frac{1}{2}B_2^{(k+1)}$$

$$C_3^{(k)} = \frac{1}{2}\left(B_4^{(k)} + B_7^{(k)} - B_3^{(k+1)}\right)$$

$$C_4^{(k)} = \frac{1}{2}\left(B_5^{(k)} + B_8^{(k)} - B_4^{(k+1)}\right)$$

$$C_5^{(k)} = \frac{1}{2}\left(B_6^{(k)} + B_{10}^{(k)} - B_5^{(k+1)}\right)$$

$$C_6^{(k)} = B_8^{(k)} - \frac{1}{2}B_7^{(k+1)}$$

$$C_7 = B_7$$

$$C_8 = A_2A_3A_4$$

$$D_2^{(k)} = C_3^{(k)} - \frac{1}{3}C_2^{(k+1)}$$

$$D_3^{(k)} = \frac{1}{3}\left(2C_2^{(k)} + C_4^{(k)} - C_3^{(k+1)}\right)$$

$$D_4^{(k)} = \frac{1}{3}\left(2C_8^{(k)} + C_5^{(k)} - C_4^{(k+1)}\right)$$

$$D_5^{(k)} = \frac{1}{3}\left(2C_8^{(k)} + C_7^{(k)} - C_6^{(k+1)}\right)$$

$$E_2^{(k)} = D_3^{(k)} - \frac{1}{4}D_2^{(k+1)}$$

$$E_3^{(k)} = \frac{1}{4}\left(3D_5^{(k)} + D_4^{(k)} - D_3^{(k+1)}\right)$$

$$F_2^{(k)} = E_3^{(k)} - \frac{1}{5}E_2^{(k+1)}$$

Order of Magnitude of Imaginary Parts of Integrals H_2, M_3, and N_3

In the detailed stability calculations the contributions of the imaginary parts of the integrals H_2, M_3, N_3, and so forth, to the function $v(c)$ are considered to be negligible in comparison with the contribution of the imaginary part of $K_1(c)$. A calculation of the orders of magnitude of I.P. $H_2(c)$, I.P. $M_3(c)$, and I.P. $N_3(c)$ from the general expressions given in the preceding pages shows that this step is justified, at least for the values of phase velocity c that appear in the stability calculations.

For example,

$$\text{I.P. } H_2(c) = \text{I.P. } H_{2_{11}}(c) = \pi A\left(w_c'\right)^2 f_1\left(y_c\right)$$

where

$$A = -\frac{1}{3T_c}\frac{c^3}{\left(w_1'\right)^3} + o\left(c^4\right) + \ldots$$

Therefore

$$\text{I.P. } H_2(c) \approx -\frac{\pi}{3} f_1\left(y_c\right)\frac{c^3}{T_1\left(w_1'\right)}$$

The contribution of I.P. $H_2(c)$ to $v(c)$ is approximately equal to $v_o\left[\frac{\alpha}{3}\frac{c^3}{T_1\left(w_1'\right)}\right]$, where $v_o = \frac{w_1'c}{T_1}$ I.P. $K_1(c)$. The quantity in the brackets is of the order of 0.03, at most, in the calculations of the present paper. (In the approximate calculations of $R_{\theta_{cr_{min}}}$ for Mach numbers very much greater than unity, c becomes large because $c > 1 - \frac{1}{M_o}$; however, α is small when c is not much greater than $1 - \frac{1}{M_o}$ and the results of the calculations of $R_{\theta_{cr_{min}}}$ based on the approximation $v(c) = \frac{w_1'c}{T_1}$ I.P. $K_1(c)$ are qualitatively correct (fig. 7).)

From the expression for $N_3(c)$, I.P. $N_3(c) \approx \frac{c^2}{2\left(w_1'\right)^2}$ I.P. $K_{11}(c)$, so that the contribution of I.P. $N_3(c)$ to $v(c)$ is approximately equal to $v_o\left[\frac{\alpha^2c^2}{2\left(w_1'\right)^2}\right]$. The quantity in brackets is of the order of 0.06 at the most.

The imaginary part of $M_3(c)$ is considerably smaller. In fact,

$$\text{I.P. } M_3(c) \approx \frac{c^6}{9T_1{}^2(w_1{}')^2} \text{ I.P. } K_1(c)$$

and the contribution of I.P. $M_3(c)$ to $v(c)$ is approximately equal to $v_o\left[\frac{-c^6\alpha^2}{9T_1{}^2(w_1{}')^2}\right]$. The quantity in brackets is of the order of 0.001 at maximum c.

APPENDIX B

CALCULATION OF MEAN-VELOCITY AND MEAN-TEMPERATURE DISTRIBUTION ACROSS BOUNDARY LAYER AND THE VELOCITY AND TEMPERATURE DERIVATIVES AT THE SOLID SURFACE

The mean-velocity and mean-temperature profiles for the several representative cases of insulated and noninsulated surfaces are calculated by a rapid approximate method that gives the slope of the velocity profiles at the surface with a maximum error of about 4 percent in the extreme case, for which $T_1 = 0.70$ and $M_o = 0.70$. The surface values of the higher velocity derivatives and the temperature derivatives required in the stability calculations are obtained directly from the equations of mean motion in terms of the calculated value of the slope of the velocity profile. The Prandtl number is taken as unity.

Mean Velocity-Temperature Distribution across Boundary Layer

In a seminar held at the California Institute of Technology in 1942, the present author has shown that a good first approximation to the mean velocity distribution across the boundary layer is obtained by assuming that the viscosity varies linearly with the absolute temperature. With this assumption, the velocity $w(\zeta)$ is the same function of the nondimensional stream function $\zeta = \dfrac{\psi^*}{\sqrt{\upsilon_o^* u_o^* x^*}}$ as in the Blasius case, and the corresponding distance from the surface $\eta = y^*\sqrt{\dfrac{u_o^*}{\upsilon_o^* x^*}}$ is obtained by a simple quadrature when $\sigma = 1$. Actually, the approximation $w(\zeta) = w_B(\zeta)$ is the first stage of an iteration process applied to the differential equations of mean motion in the laminar boundary layer, in which $\mu \propto T^{1-\epsilon}$ (ϵ is a small parameter equal to 0.24 for air), and $w(\zeta) = w_B(\zeta) + \epsilon w_1(\zeta) + \epsilon^2 w_2(\zeta) + \ldots$ Calculation of $w_1(\zeta)$ for $T_1 = 1.50$ and $T_1 = 2.00$ for $M_o \longrightarrow 0$ showed that the iteration process is rapidly convergent; the contribution of the second term to the slope of the velocity profile at the surface is 5 percent for $T_1 = 1.50$ and 8 percent for $T_1 = 2.00$. In the present calculations the maximum error in the slope introduced by taking $w(\zeta) = w_B(\zeta)$ is about 4 percent in the extreme case. (See reference 15, in which the authors make use of a linear viscosity-temperature relation. See also reference 23.)

That $w(\zeta) \equiv w_B(\zeta)$ for a linear variation of viscosity with absolute temperature is seen directly from the equations of mean motion in the laminar boundary layer. The equation of continuity is automatically satisfied by taking

$$\frac{\overline{\rho^*}}{\overline{\rho_o^*}} \overline{u^*} = \frac{\partial\psi^*}{\partial y^*}$$

and

$$\frac{\overline{\rho^*}}{\overline{\rho_o^*}} \overline{v^*} = -\frac{\partial\psi^*}{\partial x^*}$$

The stream function ψ^* and the distance along the surface x^* are selected as independent variables following the procedure of von Mises, and the dynamic equation of mean motion becomes for zero pressure gradient

$$\rho_o^{*2} \frac{\partial\overline{u^*}}{\partial x^*} = \frac{\partial}{\partial\psi^*}\left(\overline{u^*}\ \overline{\rho^*}\ \overline{u^*} \frac{\partial u^*}{\partial\psi^*}\right)$$

Define the nondimensional stream function ζ by the relation $\zeta = \dfrac{\psi^*}{\sqrt{\overline{\upsilon_o^*}\ \overline{u_o^*}\ x^*}}$. The dynamic equation takes the following form:

$$-\frac{\zeta}{2}\frac{dw}{d\zeta} = \frac{d}{d\zeta}\left(\rho\mu w \frac{dw}{d\zeta}\right)$$

Since $\rho = \dfrac{1}{T}$ in the boundary layer, if $\mu = T$, the dynamic equation in this form is identical with the equation for the isothermal Blasius flow, that is, $w(\zeta) \equiv w_B(\zeta)$, or the value of the velocity ratio w is equal to the Blasius value at the same value of ζ. The corresponding value of $\eta = y^* \sqrt{\frac{\overline{u_o^*}}{\overline{\upsilon_o^*}\ x^*}}$, the nondimensional "distance" from the surface, is obtained as follows:

$$\rho w\overline{u_o^*} = \frac{\partial \psi^*}{\partial y^*} = \sqrt{\overline{\upsilon_o^*}\ \overline{u_o^*}\ x^*}\ \frac{\partial \zeta}{\partial y}$$

or

$$\rho w = \frac{d\zeta}{d\eta}$$

$$\eta = \int_0^\zeta \frac{d\zeta}{\rho w} = \int_0^\zeta T\ \frac{d\zeta}{w}$$

If $\sigma = 1$, the energy and dynamic equations have a unique integral and

$$T = T_1 - \left[\left(T_1 - 1\right) - \frac{\gamma - 1}{2} M_o^2\right] w - \frac{\gamma - 1}{2} M_o^2 w^2$$

as shown by Crocco. Therefore,

$$\eta = T_1 \int_0^\zeta \frac{d\zeta}{w} - \left[\left(T_1 - 1\right) - \frac{\gamma - 1}{2} M_o^2\right] \zeta - \frac{\gamma - 1}{2} M_o^2 \int_0^\zeta w\ d\zeta$$

But $w(\zeta) \equiv w_B(\zeta)$, and

$$\eta = T_1 \eta_B - \left[\left(T_1 - 1\right) - \frac{\gamma - 1}{2} M_o^2\right] \int_0^{\eta_B} w_B\ d\eta_B - \frac{\gamma - 1}{2} M_o^2 \int_0^{\eta_B} w_B^2\ d\eta_B$$

The integrals $\int_0^{\eta_B} w_B \, d\eta_B$ and $\int_0^{\eta_B} {w_B}^2 \, d\eta_B$ are given in the following table, and the mean-velocity and mean-temperature profiles can be calculated rapidly by this method. (The values of $\left(\frac{\partial w}{\partial \eta}\right)_B$ are used in the approximate calculation of $R_{\theta_{cr_{min}}}$ (appendix C).)

η_B	w_B	$\zeta = \int_0^{\eta_B} w_B \, d\eta_B$	$\int_0^{\eta_B} {w_B}^2 \, d\eta_B$	$\left(\frac{\partial w}{\partial \eta}\right)_B$
0.00	0.0000	0.0000	0.0000	0.3320
.20	.0664	.0066	.0003	.3319
.40	.1328	.0265	.0024	.3314
.60	.1989	.0599	.0081	.3300
.80	.2647	.1065	.0189	.3274
1.00	.3298	.1660	.0367	.3230
1.20	.3938	.2385	.0630	.3165
1.40	.4563	.3236	.0993	.3079
1.60	.5168	.4210	.1468	.2967
1.80	.5748	.5302	.2064	.2825
2.00	.6298	.6508	.2792	.2663
2.20	.6813	.7821	.3654	.2483
2.40	.7290	.9231	.4648	.2280
2.60	.7725	1.0733	.5776	.2064
2.80	.8115	1.2319	.7034	.1835
3.00	.8460	1.3978	.8411	.1618
3.20	.8761	1.5702	.9897	.1408
3.40	.9018	1.7480	1.1478	.1180
3.60	.9233	1.9306	1.3145	.0986
3.80	.9411	2.1171	1.4884	.0805
4.00	.9555	2.3067	1.6682	.0640
4.40	.9759	2.6933	2.0419	
4.80	.9878	3.0863	2.4280	
5.20	.9942	3.4828	2.8211	
5.60	.9975	3.8812	3.2180	
6.00	.9990	4.2805	3.6167	

With the approximation that μ varies linearly with the absolute temperature, the slope of the velocity profile at the solid surface is simply related to the slope of the Blasius profile. Thus

$$\frac{\partial w}{\partial \eta} = \frac{dw}{d\zeta}\frac{d\zeta}{d\eta} = \rho w \frac{dw}{d\zeta}$$

Since $w(\zeta) \equiv w_B(\zeta)$,

$$\frac{\partial w}{\partial \eta} = \rho\left(\frac{\partial w}{\partial \eta}\right)_B$$

and

$$\left(\frac{\partial w}{\partial \eta}\right)_1 = \frac{0.332}{T_1}$$

or

$$\frac{dw}{dy} = w_1{}' = \frac{0.332}{T_1} b$$

where b is the value of η at the "edge" of the boundary layer (when w reaches an arbitrarily prescribed value close to unity). It is seen that the shear stress at the surface (or the skin friction) has the same value as in the Blasius case

$$\overline{\tau_1^*} = \overline{\mu_1^*}\left(\frac{\partial \overline{u^*}}{\partial y^*}\right)_1 = \overline{\mu_1^*}\ \overline{u_o^*}\left(\frac{\partial w}{\partial \eta}\right)_1 \frac{\partial \eta}{\partial y} = \overline{\mu_o^*}\ \mu_1\ \overline{u_o^*}\ \frac{\partial \eta}{\partial y}\left[\rho_1\left(\frac{\partial w}{\partial \eta}\right)_{1_B}\right] = \left(\overline{\tau_1^*}\right)_B$$

The reliability of this approximation can be judged from the calculations of the skin-friction coefficient in reference 24, in which $\mu \propto T^{0.76}$. From figure 2 of reference 24, the value of the skin-friction coefficient for an insulated surface at a Mach number of 3.0 $\left(T_1 = 2.823\right)$ is only 12 percent lower than the Blasius value and only 2 percent lower at a Mach number of 2.0 $\left(T_1 = 1.81\right)$. For the noninsulated surface, with $T_1 = 0.25$, the value of the skin-friction coefficient at $M_o = 0$ is only 7 percent greater than the Blasius value and 12 percent greater at a Mach number of 3.00.

Since the shear stress at the surface is unchanged in first approximation, the boundary-layer momentum thickness has the same value as for the Blasius flow

$$\theta\sqrt{\frac{\overline{u_o^*}}{\overline{\upsilon_o^*}\ x^*}} = 0.6667$$

The expression for the displacement thickness δ^* gives a measure of the effect of the thermal conditions at the solid surface and the free-stream Mach number on the thickness of the boundary layer. By definition,

$$\delta^*\sqrt{\frac{\overline{u_o^*}}{\overline{\upsilon_o^*}\ x^*}} = \int_0^\infty (1 - \rho w)\ d\eta$$

From the relation between $d\eta$ and $d\eta_B$

$$\delta^*\sqrt{\frac{\overline{u_o^*}}{\overline{\upsilon_o^*}\ x^*}} = \int_0^\infty \left[(T - 1) + \left(1 - w_B\right)\right] d\eta_B$$

$$= 1.73 + \left(T_1 - 1\right)1.73 + \frac{\gamma - 1}{2} M_o^{\ 2}(0.6667)$$

$$= 1.73\ T_1 + \frac{\gamma - 1}{2} M_o^{\ 2}(0.6667)$$

For the Blasius flow

$$\left(\delta^*\sqrt{\frac{u_o^*}{\nu_o^*\ x^*}}\right)_B = 1.730$$

The "thickness" of the boundary layer b is given by

$$b = 5.60 + \left(T_1 - 1\right)1.73 + \frac{\gamma - 1}{2} M_o^2(0.6667)$$

and the form parameter $H = \dfrac{\delta^*}{\theta}$ is

$$H = 2.50\ T_1 + \frac{\gamma - 1}{2} M_o^2$$

For the insulated surface,

$$H = 2.50 + 3.50\left(\frac{\gamma - 1}{2} M_o^2\right)$$

Calculation of Mean-Velocity and Mean-Temperature Derivatives

Because of the sensitivity of the stability characteristics of the laminar boundary layer to the behavior of the quantity $\dfrac{d}{dy}\left(\rho\dfrac{dw}{dy}\right)$, the values of the required velocity and temperature derivatives at the surface are calculated directly from the equations of mean motion, with $\mu = T^m$ $(m = 0.76$ for air). Now $\dfrac{d\zeta}{dy} = \zeta' = b\rho w$ so that the dynamic equation is $-b\dfrac{\zeta}{2}w' = \left(T^m w'\right)'$. Since $\zeta(0) = \zeta'(0) = 0$,

$$w_1'' = -\frac{m}{T_1} T_1' w_1'$$

where $T_1' = w_1' \left[\frac{\gamma - 1}{2} M_o^2 - \left(T_1 - 1\right)\right]$, if $\sigma = 1$. In other words, the value of w_1'' is readily computed from the value of w_1'. In general, $w_1^{(k)}$ is determined from the relation

$$-\frac{b}{2} (\zeta w')^{k-2} = \left(T^m w'\right)^{k-1}$$

or

$$w_1^{(k)} = -\left[(k-1) \frac{mT_1'}{T_1} w_1^{(k-1)} + \frac{(k-1)!}{(k-3)!2!} \frac{\left(T^m\right)_1''}{T_1^m} w_1^{(k-2)} + \ldots\right.$$

$$+ \frac{(k-1)!}{(k-1-s)s!} \frac{\left(T^m\right)_1^{(s)}}{T_1^m} w_1^{(k-s)} + \ldots + \left.\frac{\left(T^m\right)_1^{(k-1)}}{T_1^m} w_1'\right]$$

$$- \frac{b}{2T_1^m}\left[\zeta_1^{(k-2)} w_1' + (k-2)\zeta_1^{(k-3)} w_1'' + \ldots\right.$$

$$\left. + \frac{(k-2)!}{(k-2-r)!r!} \zeta^{(k-2-r)} w_1^{(r+1)} + \ldots + \frac{(k-3)(k-2)}{2} \zeta_1'' w_1^{(k-3)}\right]$$

where

$$\zeta_1^{(p)} = b(\rho w)_1^{(p-1)}$$

$$= b\left[\rho^{(p-1)} w + (p - 1)\rho_1^{(p-2)} w_1' + \ldots \right.$$

$$\left. + \frac{(p - 1)!}{(p - q - 1)!q!}\rho_1^{(p-q-1)} w_1^{(q)} + \ldots + \rho_1 w_1^{(p-1)}\right]$$

$$p = 1, 2, \ldots 5$$

and

$$\rho_1 = \frac{1}{T_1}$$

$$\rho_1' = -\frac{T_1'}{T_1^{\,2}}$$

$$\rho_1'' = \frac{2\left(T_1'\right)^2}{T_1^{\,3}} - \frac{T_1''}{T_1^{\,2}}$$

$$\rho_1''' = 6T_1' \frac{T_1''}{T_1^{\,3}} - 6\,\frac{\left(T_1'\right)^3}{T_1^{\,4}} - \frac{T_1'''}{T_1^{\,2}}$$

$$\frac{\left(T^m\right)''}{T_1^{\,m}} = m(m - 1)\,\frac{\left(T_1'\right)^2}{T_1^{\,2}} + m\,\frac{T_1''}{T_1}$$

$$\frac{\left(T^m\right)'''}{T_1{}^m} = (m,\ 2)\ \frac{\left(T_1{}'\right)^3}{T_1{}^3} + 3m(m - 1)\ \frac{T_1{}'T_1{}''}{T_1{}^2} + m\ \frac{T_1{}'''}{T_1}$$

$$\frac{\left(T^m\right)^{iv}}{T_1{}^m} = (m,\ 3)\ \frac{\left(T_1{}'\right)^4}{T_1{}^4} + 6(m,\ 2)\ \frac{\left(T_1{}'\right)^2T_1{}''}{T_1{}^3} + \frac{m(m-1)}{T_1{}^2}\ \Big[4T_1{}'T_1{}'''$$

$$+ 3\left(T_1{}''\right)^2\Big] + m\ \frac{T_1{}^{iv}}{T_1}$$

$$\frac{\left(T^m\right)^{v}}{T_1{}^m} = (m,\ 4)\ \frac{\left(T_1{}'\right)^5}{T_1{}^5} + 10(m,\ 3)\ \frac{\left(T_1{}'\right)^3T_1{}''}{T_1{}^4} + \frac{(m,\ 2)}{T_1{}^3}\ \Big[15\left(T_1{}'\right)\left(T_1{}''\right)^2$$

$$+ 10\left(T_1{}'\right)^2T_1{}'''\Big] + \frac{m(m-1)}{T_1{}^2}\ \Big[10T_1{}''T_1{}''' + 5T_1{}'T_1{}^{iv}\Big] + m\ \frac{T_1{}^{v}}{T_1}$$

$$\frac{\left(T^m\right)^{vi}}{T_1{}^m} = (m,\ 5)\ \frac{\left(T_1{}'\right)^6}{T_1{}^6} + 15(m,\ 4)\ \frac{\left(T_1{}'\right)^4T_1{}''}{T_1{}^5} + \frac{(m,\ 3)}{T_1{}^4}\ \Big[45\left(T_1{}'\right)^2\left(T_1{}''\right)^2$$

$$+ 20\left(T_1{}'\right)^3T_1{}'''\Big] + \frac{(m,\ 2)}{T_1{}^3}\Big[60T_1{}'T_1{}''T_1{}''' + 15\left(T_1{}'\right)^2T_1{}^{iv} + 15\left(T_1{}''\right)^3\Big]$$

$$+ \frac{m(m - 1)}{T_1{}^2}\ \Big[10\left(T_1{}'''\right)^2 + 15T_1{}''T_1{}^{iv} + 6T_1{}'T_1{}^{v}\Big] + m\ \frac{T_1{}^{vi}}{T_1}$$

$$(m,\ 2) = m(m - 1)(m - 2)$$

$$(m,\ n) = m(m - 1)(m - 2)\ .\ .\ .\ (m - n)$$

$$m = 0.76$$

$$(m,\ 1) = -0.1824$$

$$(m,\ 2) = 0.226176$$

$$(m,\ 3) = -0.506634$$

$$(m,\ 4) = 1.641495$$

$$(m,\ 5) = -6.959939$$

$$T_1' = aw_1'$$

where

$$a = \frac{\gamma - 1}{2} M_o^{\ 2} - \left(T_1 - 1\right)$$

$$T_1'' = aw_1'' - (\gamma - 1)M_o^{\ 2}\left(w_1'\right)^2$$

$$T_1''' = aw_1''' - 3(\gamma - 1)M_o^{\ 2}w_1'w_1''$$

$$T_1^{iv} = aw_1^{iv} - (\gamma - 1)M_o^{\ 2}\left[3\left(w_1''\right)^2 + 4w_1'w_1'''\right]$$

$$T_1^{v} = aw_1^{v} - 5(\gamma - 1)M_o^2\left(2w_1''w_1''' + w_1'w_1^{iv}\right)$$

$$T_1^{vi} = aw_1^{vi} - (\gamma - 1)M_o^2\left[10\left(w_1'''\right)^2 + 15w_1''w_1^{iv} + 6w_1'w_1^{v}\right]$$

Each velocity derivative is determined from the knowledge of all the preceding derivatives.

APPENDIX C

RAPID APPROXIMATION TO THE FUNCTION $(1 - 2\lambda)v(c)$ AND THE MINIMUM CRITICAL REYNOLDS NUMBER

In section 5, a criterion was derived for the dependence of the minimum critical Reynolds number $R_{\theta_{cr_{min}}}$ on the local distribution of mean velocity and mean temperature across the boundary layer. It was found that

$$R_{\theta_{cr_{min}}} \approx \frac{6}{T_1} \frac{\left[T\left(c_o\right)\right]^{1.76}}{c_o^{4}\sqrt{1 - M_o^{2}\left(1 - c_o\right)^{2}}}$$

where c_o is the value of c for which $(1 - 2\lambda)v(c) = 0.580$ and

$$v(c) = -\pi \frac{w_1'c}{T_1}\left[\frac{T^2}{(w')^3} \frac{d}{dy}\left(\frac{w'}{T}\right)\right]_{w=c}$$

$$= \frac{-\pi\left(\frac{\partial w}{\partial \eta}\right)_1 c}{T_1}\left[\frac{T^2}{\left(\frac{\partial w}{\partial \eta}\right)^3} \frac{\partial}{\partial \eta}\left(\frac{1}{T} \frac{\partial w}{\partial \eta}\right)\right]_{w=c}$$

$$\lambda(c) = \frac{w_1'\left(y_c - y_1\right)}{c} - 1$$

$$= \frac{\eta\left(\frac{\partial w}{\partial \eta}\right)_1}{c} - 1$$

A rapid method for the calculation of the function $(1 - 2\lambda)v(c)$ and the minimum critical Reynolds number is developed by making use of the approximation that the viscosity varies linearly with the absolute temperature (appendix B). (Since the effect of variable viscosity on the mean-velocity profile is overestimated in this approximation, the values of $R_{\theta_{cr_{min}}}$ (fig. 6(a)) calculated by this method are lower than the values calculated for $\mu = T^{0.76}$ when heat is added to the fluid through the solid surface and higher when heat is withdrawn from the fluid.)

For $\mu = T$, the dynamic equation (appendix B) is

$$-\frac{\zeta}{2}\frac{\partial w}{\partial \eta} = \frac{\partial}{\partial \eta}\left(\frac{1}{\rho}\frac{\partial w}{\partial \eta}\right)$$

and therefore

$$\frac{T^2}{\frac{\partial w}{\partial \eta}}\frac{\partial}{\partial \eta}\left(\frac{1}{T}\frac{\partial w}{\partial \eta}\right) = T\left(\frac{\frac{\partial^2 w}{\partial \eta^2}}{\frac{\partial w}{\partial \eta}} - \frac{1}{T}\frac{\partial T}{\partial \eta}\right)$$

$$= -\left(\frac{\zeta}{2} + 2\frac{\partial T}{\partial \eta}\right)$$

But

$$\frac{\partial T}{\partial \eta} = \frac{1}{T}\left(\frac{\partial T}{\partial \eta}\right)_B$$

so that

$$\frac{T^2}{\frac{\partial w}{\partial \eta}}\frac{\partial}{\partial \eta}\left(\frac{1}{T}\frac{\partial w}{\partial \eta}\right) = -\left[\frac{\zeta}{2} + \frac{2}{T}\left(\frac{\partial T}{\partial \eta}\right)_B\right]$$

where

$$T = T_1 - \left[\left(T_1 - 1\right) - \frac{\gamma - 1}{2} M_o^2\right] w_B - \frac{\gamma - 1}{2} M_o^2 w_B^2$$

$$\left(\frac{\partial T}{\partial \eta}\right)_B = - \left[\left(T_1 - 1\right) - \frac{\gamma - 1}{2} M_o^2\left(1 - 2w_B\right)\right]\left(\frac{\partial w}{\partial \eta}\right)_B$$

Finally,

$$v(c) = + \frac{0.332\pi}{T_1^2} \left\{\left[\frac{w_B T^2}{\left(\frac{\partial w}{\partial \eta}\right)_B^2} \left[\frac{\zeta}{2} + \frac{2}{T}\left(\frac{\partial T}{\partial \eta}\right)_B\right]\right]_{w_B=c}\right\}$$

The required values of w_B, $\left(\frac{\partial w}{\partial \eta}\right)_B$, and ζ are obtained from the table in appendix B.

The small correction to the slope $\lambda(c)$ is easily calculated once the mean velocity profile has been obtained (appendix B). Thus

$$\lambda(c) = \frac{0.332}{T_1} \frac{\eta}{c} - 1 \qquad , \qquad \lambda(0) = 0$$

The quantity $(1 - 2\lambda)v(c)$ has been calculated as a function of c for various values of T_1 at $M_o = 0$, 0.70, 1.30, 1.50, 2.00, 3.00, and 5.00, and the results of these calculations are given in the following table. The decisive stabilizing influence of withdrawing heat from the fluid at supersonic velocities is illustrated in figure 7.

T_1	c_o	$R_{\theta_{cr_{min}}}$
$M_o = 0$		
0.70	0.1945	3650
.80	.2695	1080
.90	.3485	402
1.25	.5435	67
1.50	.6240	36
$M_o = 1.30$; $c > 0.231$		
0.90	0.2455	9230
1.05	.4075	392
1.20	.5170	121
1.3422	.5450	92
1.50	.6355	42
$M_o = 2.00$; $c > 0.500$		
1.63	0.5074	671
1.65	.5438	207
1.70	.6155	75
1.75	.6749	40
1.81	.7275	25
1.85	.7612	19
$M_o = 5.00$; $c > 0.800$		
5.19	0.8008	174
5.20	.8036	80
5.30	.8262	23
5.75	.9008	6
6.0625	.9350	3

T_1	c_o	$R_{\theta_{cr_{min}}}$
$M_o = 0.70$		
0.70	0.1670	8440
.80	.2390	2110
.90	.3265	613
1.25	.5425	74
1.50	.6265	38
$M_o = 1.50$; $c > 0.333$		
1.30	0.3450	2770
1.35	.4585	275
1.40	.5505	99
1.4556	.6276	49
1.60	.7732	16
$M_o = 3.00$; $c > 0.667$		
2.48	0.6730	186
2.52	.7058	59
2.62	.7655	24
2.72	.8105	14
2.77	.8295	10
2.8225	.8500	9

APPENDIX D

BEHAVIOR OF $\frac{d}{dy}\left(\rho \frac{dw}{dy}\right)$ FROM EQUATIONS OF MEAN MOTION

In order to determine the effect of free-stream Mach number, thermal conditions at the solid surface, or free-stream pressure gradient on laminar stability, it is necessary to know the relation between these physical parameters and the distribution of the quantity $\rho \frac{dw}{dy}$ across the boundary layer. The value of $\frac{d}{dy}\left(\rho \frac{dw}{dy}\right)$ at the solid surface is obtained directly from the dynamic equation (equations (6.3) and (7.2)). The value of $\frac{d^2}{dy^2}\left(\rho \frac{dw}{dy}\right)$ at the surface, which is also useful in the discussion of laminar stability, is obtained from the dynamic and energy equations as follows:

$$\left[\frac{d^2}{dy^2}\left(\rho \frac{dw}{dy}\right)\right]_1 = \left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1$$

$$= \frac{w_1'''}{T_1} - \frac{2w_1''T_1'}{T_1^2} - \frac{w_1'T_1''}{T_1^2} + \frac{2w_1'(T_1')^2}{T_1^3}$$

Differentiating the dynamic equation once yields the result

$$w_1''' = -\frac{2mT_1'w_1''}{T_1} - w_1'\left[m(m-1)\frac{(T_1')^2}{T_1^2} + m\frac{T_1''}{T_1}\right]$$

At the solid surface the rate of change of temperature $\frac{d\overline{T^*}}{dt^*}$ and the rate at which the work is done by pressure gradient $\overline{u^*}\,\frac{\partial\overline{p^*}}{\partial x^*}$ both vanish, and the rate at which a fluid element loses heat by conduction equals the rate at which mechanical energy is transformed into heat by viscous dissipation. The energy equation becomes

$$-\left[\frac{\partial}{\partial y^*}\left(\overline{k^*}\,\frac{\partial\overline{T^*}}{\partial y^*}\right)\right]_1 = \overline{\mu_1^*}\left(\frac{\partial\overline{u^*}}{\partial y^*}\right)_1^2$$

or

$$T_1'' = -\sigma(\gamma - 1)M_o^2(w_1')^2 - m\,\frac{(T_1')^2}{T_1} < 0$$

Utilizing the expression for w_1''' and T_1'' gives

$$\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1 = -2(m + 1)\,\frac{T_1'}{T_1}\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_1 + \sigma(1 + m)(\gamma - 1)M_o^2\,\frac{(w_1')^3}{T_1^2}$$

where

$$\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_1 = -\frac{m + 1}{T_1^2}T_1'w_1' - \frac{1}{T_1^{1+m}}\,\frac{\delta^2}{\overline{\upsilon_o^*}}\,\frac{d\overline{u_o^*}}{dx^*}$$

From this expression for $\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1$ the following conclusions, which are utilized in the stability analysis, are reached:

When $\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_1$ vanishes, the quantity $\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1$ is still positive.

When the free-stream velocity is uniform,

$$\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1 = \sigma(1 + m)(\gamma - 1)M_o^{\,2} \frac{(w_1')^3}{T_1^{\,2}} + 2(1 + m)^2 \frac{(T_1')^2}{T_1^{\,3}} w_1';$$

that is, $\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1$ is always positive.

When the surface is insulated,

$$\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1 = \sigma(1 + m)(\gamma - 1)M_o^{\,2} \frac{(w_1')^3}{T_1^{\,2}}$$

and $\left[\frac{d^2}{dy^2}\left(\frac{w'}{T}\right)\right]_1$ is always positive, regardless of the pressure gradient.

APPENDIX E

CALCULATION OF CRITICAL MACH NUMBER FOR STABILIZATION OF LAMINAR BOUNDARY LAYER

For thermal equilibrium the rate of heat conduction from the gas to the solid surface balances the rate at which heat is radiated from the surface. If the rate at which heat is withdrawn from the fluid reaches or exceeds a certain critical value at a given local supersonic Mach number, the laminar boundary-layer flow is stable at all Reynolds numbers. (See section 6b.) The purpose of the following brief calculation is to determine the equilibrium surface temperatures at several Mach numbers and compare these temperatures with the critical temperatures for laminar stability. (See fig. 8.)

When the solid surface is in thermal equilibrium

$$\int_0^L \overline{k_1^*}\left(\frac{\partial \overline{T^*}}{\partial y^*}\right)_1 dx = \int_0^L \epsilon\bar{\sigma}\left[\left(\overline{T_1^*}\right)^4 - \left(\overline{T_o^*}\right)^4\right] dx \qquad (1)$$

where ϵ is the emissivity, $\bar{\sigma}$ is the Boltzmann constant, and the other symbols have already been defined. (See references 14 and 15.) Consider the case in which the free stream is uniform and the temperature is constant along the surface. For $\sigma = 1$,

$$\left(\frac{\partial \overline{T^*}}{\partial y^*}\right)_1 = \frac{\overline{T_o^*}}{\delta}\left(T_s - T_1\right)\left(\frac{\partial w}{\partial y}\right)_1$$

where stagnation temperature T_s equals $1 + \frac{\gamma - 1}{2} M_o^2$.

Also $\left(\frac{\partial w}{\partial y}\right)_1 = \delta\sqrt{\frac{\overline{\mu_o^*}}{\overline{\upsilon_o^*}\ x^*}}\ \frac{0.332}{T_1}$ if the approximation $\mu = T$ is employed. (See appendix B.) Since $\overline{k_1^*} = c_p\overline{\mu_o^*}\mu_1 = c_p\overline{\mu_o^*}T_1$,

$$\overline{k_1^*}\left(\frac{\partial \overline{T^*}}{\partial y^*}\right)_1 = 0.332\ c_p\sqrt{\overline{\mu_o^*}\ \overline{\rho_o^*}\ \overline{a_o^*}}\ \overline{T_o^*}\left(T_s - T_1\right)\sqrt{M_o}\ \frac{1}{\sqrt{x^*}}$$

When the integrations in equation (1) are carried out, the following relation is obtained for the determination of the equilibrium surface temperature:

$$\sqrt{K}\left(T_1^{\,4} - 1\right) = \left(T_s - T_1\right)\sqrt{M_o}$$

where

$$K = 2.27\ \frac{\left(\overline{T_o^*}\right)^6\ \epsilon^2\bar{\sigma}^2 L}{c_p^{\,2}\ \overline{\rho_o^*}\ \overline{\mu_o^*}\sqrt{(\gamma - 1)c_p\overline{T_o^*}}}$$

The equilibrium surface temperature under free-flight conditions is affected principally by the variation in density $\overline{\rho_o^*}$ with altitude h. The results of calculations carried out for altitudes of 50,000 and 100,000 feet are given in the following table:

h (ft)	M_o	$T_s - T_{1_{equil}}$	$T_s - T_{1_{cr}}$ (fig. 8)
50×10^3	3.0	0.370	0.355
100×10^3	2.0	.220	.185

In these calculations the following data are used:

$$\epsilon^2 = 0.50$$

$$L = 2 \text{ ft}$$

$$\overline{T_o^*} = 400^\circ \text{ F abs.}$$

$$\overline{\sigma} = 4.80 \times 10^{-13} \text{ Btu/sec/ft}^2/(\text{deg F abs.})^4$$

$$c_p = 7.73 \text{ Btu/slug/deg F abs.}$$

$$\overline{\mu_o^*} = 3.02 \times 10^{-7} \text{ slugs/ft-sec}$$

$$\overline{a_o^*} = 980 \text{ ft/sec}$$

$$\overline{\rho_o^*} = 3.61 \times 10^{-4} \text{ slugs/ft}^3 \text{ at } 50{,}000 \text{ ft}$$

$$= 3.31 \times 10^{-5} \text{ slugs/ft}^3 \text{ at } 100{,}000 \text{ ft}$$

$$K = 3.35 \times 10^{-4} \text{ at } 50{,}000 \text{ ft}$$

$$= 3.66 \times 10^{-3} \text{ at } 100{,}000 \text{ ft}$$

Since $T_s - T_{1_{equil}} > T_s - T_{1_{cr}}$ for $M_o \approx 3$ at 50,000 feet altitude and for $M_o \approx 2$ at 100,000 feet altitude, the laminar boundary layer is completely stable under these conditions.

It should be noted that under wind-tunnel-test conditions in which the model is stationary, these radiation-conduction effects are absent, not only because of reradiation from the walls of the wind tunnel but also because the surface temperatures are low - generally of the order of room temperature.

REFERENCES

1. Heisenberg, Werner: Über Stabilität und Turbulenz von Flüssigkeitsströmen. Ann. d. Phys., Vierte Folge, Bd. 74, No. 15, 1924, pp. 577-627.

2. Tollmien, W.: Über die Entstehung der Turbulenz. Nachr. d. Ges. d. Wiss. zu Göttingen, Math.-Phys. Kl., 1929, pp. 21-44. (Available as NACA TM No. 609, 1931.)

3. Tollmien, W.: Ein allgemeines Kriterium der Instabilität laminarer Geschwindigkeitsverteilungen. Nachr. d. Ges. d. Wiss. zu Göttingen, Math.-Phys. Kl., Neue Folge, Bd. 1, Nr. 5, 1935, pp. 79-114. (Available as NACA TM No. 792, 1936.)

4. Schlichting. H.: Amplitudenverteilung und Energiebilanz der kleinen Störungen bei der Plattenströmung. Nachr. d. Ges. Wiss. zu Göttingen, Math.-Phys. Kl., Neue Folge, Bd. 1, Nr. 4, 1935, pp. 47-78.

5. Lin, C. C.: On the Stability of Two-Dimensional Parallel Flows. Part I. Quarterly Appl. Math., vol. III, no. 2, July 1945, pp. 117-142; Part II, vol. III, no. 3, Oct. 1945, pp. 218-234; and Part III, vol. III, no. 4, Jan. 1946, pp. 277-301.

6. Liepmann, Hans W.: Investigations on Laminar Boundary-Layer Stability and Transition on Curved Boundaries. NACA ACR No. 3H30, 1943.

7. Schubauer, G. B., and Skramstad, H. K.: Laminar-Boundary-Layer Oscillations and Transition on a Flat Plate. NACA ACR, April 1943.

8. Lees, Lester, and Lin, Chia Chiao: Investigation of the Stability of the Laminar Boundary Layer in a Compressible Fluid. NACA TN No. 1115, 1946.

9. Pretsch, J.: Die Stabilität der Laminarströmung bei Druckgefälle und Druckanstieg. Forschungsbericht Nr. 1343, Deutsche Luftfahrtforschung (Göttingen), 1941.

10. Oswatitsch, K., and Wieghardt, K.: Theoretische Untersuchungen über stationäre Potentialströmungen und Grenzschichten bei hohen Geschwindigkeiten. Bericht S13/1. Teil der Lilienthal-Gesellschaft für Luftfahrtforschung, 1942, pp. 20-22.

11. Tietjens, O.: Beiträge zur Entstehung der Turbulenz. Z.f.a.M.M., Bd. 5, Heft 3, June 1925, pp. 200-217.

12. Prandtl. L.: The Mechanics of Viscous Fluids. The Development of Turbulence. Vol. III of Aerodynamic Theory, div. G, sec. 26, W. F. Durand, ed., Julius Springer (Berlin), 1935, pp. 178-190.

13. Synge, J. L.: Hydrodynamical Stability. Semicentennial Publications, American Math. Soc.; vol. II: Semicentennial Addresses, 1938, pp. 255-263.

14. Kiebel, I. A.: Boundary Layer in Compressible Liquid with Allowance for Radiation. Comptes Rendus Acad. Sci. USSR, vol. XXV, no. 4, 1939, pp. 275-279.

15. Hantzsche, W., and Wendt, H.: Die laminare Grenzschicht der ebenen Platte mit und ohne Wärmeübergang unter Berücksichtigung der Kompressibilität. Jahrb. 1942 der deutschen Luftfahrtforschung, R. Oldenbourg (Munich) pp. I 40 - I 50.

16. Liepmann, H. W.: Investigation of Boundary Layer Transition on Concave Walls. NACA ACR No. 4J28, 1945, pp. 14-20.

17. Landau, L.: On the Problem of Turbulence. Comptes Rendus Acad. Sci. USSR, vol. XLIV, no. 8, 1944, pp. 311-314.

18. Pretsch, J.: Die Anfachung instabiler Störungen in einer laminaren Reibungsschicht. Jahrb. 1942 der deutschen Luftfahrtforschung, R. Oldenbourg (Munich), pp. I 54 - I 71.

19. Liepmann, Hans W., and Fila, Gertrude H: Investigations of Effects of Surface Temperature and Single Roughness Elements on Boundary-Layer Transition. NACA TN No. 1196, 1947.

20. Frick, Charles W., Jr., and McCullough, George B.: Tests of a Heated Low-Drag Airfoil. NACA ACR, Dec. 1942.

21. Schlichting, H., and Ulrich, A.: Zur Berechnung des Umschlages laminar turbulent. Jahrb. 1942 der deutschen Luftfahrtforschung, R. Oldenbourg (Munich), pp. I 8 - I 35.

22. Dorodnitzyn, A.: Laminar Boundary Layer in Compressible Fluid. Comptes Rendus Acad. Sci. USSR, vol. XXXIV, no. 8, 1942, pp. 213-219.

23. Brainerd, J. G., and Emmons, H. W.: Effect of Variable Viscosity on Boundary Layers, with a Discussion of Drag Measurements. Jour. Appl. Mech., vol. 9, no. 1, March 1942, pp. A-1 - A-6.

24. von Kármán, Th., and Tsien, H. S.: Boundary Layer in Compressible Fluids. Jour. Aero. Sci., vol. 5, no. 6, April 1938, pp. 227-232.

TABLE I

AUXILIARY FUNCTIONS FOR CALCULATING THE STABILITY OF THE LAMINAR

BOUNDARY LAYER FOR INSULATED SURFACE

c	λ	v	L	H_1	H_2	M_3	N_3
$M_0 = 0$							
0.0372	0.0000	0.0004	0.0102	0.5220	0.2889	0.0689	0.2999
.0744	.0001	.0029	.0285	.4748	.2740	.0604	.3064
.1115	.0003	.0099	.0561	.4303	.2590	.0530	.3124
.1486	.0006	.0235	.0940	.3887	.2433	.0460	.3161
.1857	.0012	.0462	.1430	.3499	.2278	.0403	.3211
.2226	.0021	.0802	.2040	.3139	.2120	.0350	.3230
.2594	.0033	.1284	.2782	.2808	.1958	.0301	.3217
.2960	.0050	.1937	.3670	.2505	.1797	.0256	.3174
.3323	.0071	.2794	.4721	.2232	.1639	.0217	.3084
.3682	.0098	.3896	.5960	.1987	.1487	.0180	.2935
.4037	.0131	.5286	.7418	.1770	.1350	.0139	.2708
.4143	.0142	.5767	.7904	.1711	.1312	.0125	.2618
$M_0 = 0.50$							
0.0362	-0.0000	-0.0004	-0.0148	0.5122	0.2223	0.0443	0.1927
.0723	-.0000	-.0001	-.0234	.4671	.2127	.0401	.2086
.1085	.0001	.0029	-.0244	.4246	.2019	.0356	.2193
.1446	.0003	.0107	-.0169	.3847	.1904	.0316	.2280
.1806	.0007	.0254	-.0003	.3474	.1789	.0282	.2366
.2166	.0014	.0492	.0260	.3127	.1662	.0249	.2420
.2525	.0023	.0846	.0627	.2807	.1530	.0217	.2425
.2882	.0036	.1342	.1103	.2513	.1390	.0188	.2406
.3237	.0054	.2010	.1695	.2246	.1247	.0158	.2333
.3588	.0076	.2882	.2412	.2005	.1104	.0128	.2179
.3936	.0103	.4000	.3261	.1790	.0963	.0094	.1914
.4280	.0137	.5407	.4247	.1602	.0828	.0055	.1444
.4306	.0140	.5526	.4327	.1589	.0816	.0051	.1397
.4362	.0146	.5794	.4501	.1560	.0792	.0038	.1262
$M_0 = 0.70$							
0.0353	-0.0000	-0.0009	-0.0321	0.5031	0.1839	0.0321	0.1484
.0705	-.0000	-.0024	-.0590	.4599	.1786	.0300	.1652
.1058	-.0000	-.0025	-.0791	.4191	.1721	.0279	.1819
.1410	.0001	.0006	-.0914	.3808	.1652	.0257	.1981
.1762	.0004	.0090	-.0951	.3448	.1569	.0233	.2128
.2114	.0008	.0248	-.0896	.3113	.1478	.0209	.2259
.2464	.0015	.0501	-.0741	.2802	.1379	.0187	.2358
.2813	.0026	.0872	-.0478	.2516	.1272	.0165	.2436
.3161	.0039	.1389	-.0098	.2255	.1157	.0142	.2466
.3505	.0058	.2082	.0412	.2018	.1042	.0118	.2417
.3847	.0081	.2985	.1067	.1806	.0925	.0085	.2272
.4185	.0109	.4137	.1886	.1619	.0813	.0052	.1987
.4352	.0126	.4821	.2363	.1534	.0760	.0030	.1787
.4452	.0137	.5270	.2674	.1486	.0733	.0016	.1618
.4559	.0149	.5790	.3027	.1436	.0709	-.0002	.1575

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE I - Concluded

AUXILIARY FUNCTIONS FOR CALCULATING THE STABILITY OF THE LAMINAR

BOUNDARY LAYER FOR INSULATED SURFACE - Concluded

c	λ	v	L	H_1	H_2	M_3	N_3
$M_o = 0.90$							
0.0334	0.0000	-0.0015	-0.0503	0.4816	0.1303	0.0180	0.0908
.0667	-.0001	-.0047	-.0972	.4422	.1298	.0185	.1133
.1001	-.0002	-.0082	-.1389	.4049	.1281	.0185	.1366
.1335	-.0002	-.0102	-.1746	.3696	.1253	.0182	.1594
.1669	-.0001	-.0090	-.2034	.3365	.1213	.0175	.1825
.2002	.0001	-.0029	-.2250	.3054	.1163	.0166	.2055
.2335	.0006	.0098	-.2387	.2765	.1103	.0157	.2252
.2666	.0012	.0312	-.2441	.2497	.1030	.0143	.2439
.2997	.0022	.0634	-.2407	.2251	.0947	.0128	.2597
.3326	.0034	.1086	-.2281	.2026	.0855	.0110	.2703
.3652	.0051	.1697	-.2063	.1823	.0759	.0090	.2674
.3976	.0072	.2496	-.1730	.1641	.0656	.0060	.2515
.4296	.0098	.3518	-.1302	.1480	.0560	.0021	.2185
.4612	.0130	.4805	-.0784	.1340	.0464	-.0036	.1431
.4636	.0132	.4913	-.0744	.1330	.0463	-.0040	.1373
.4812	.0153	.5788	-.0421	.1261	.0418	-.0076	.1004
$M_o = 1.10$							
0.0990	-0.0003	-0.0140	-0.2037	0.4026	0.0673	0.0012	0.0806
.1320	-.0004	-.0206	-.2630	.3682	.0686	.0038	.1068
.1650	-.0005	-.0255	-.3166	.3358	.0683	.0051	.1319
.1980	-.0004	-.0272	-.3640	.3054	.0667	.0058	.1598
.2309	-.0002	-.0232	-.4049	.2770	.0632	.0064	.1864
.2638	.0002	-.0125	-.4396	.2506	.0581	.0062	.2101
.2965	.0009	.0072	-.4680	.2263	.0516	.0058	.2293
.3292	.0018	.0382	-.4906	.2040	.0431	.0047	.2416
.3616	.0031	.0829	-.5086	.1837	.0333	.0031	.2454
.3938	.0049	.1442	-.5239	.1655	.0218	.0005	.2310
.4246	.0097	.2247	-.5516	.1498	.0081	-.0032	.1834
.4572	.0098	.3300	-.5675	1350	-.0060	-.0087	0764
.4836	.0126	.4407	-.6112	.1245	-.0203	-.0157	- 0737
.5104	.0160	.5789	-.6875	.1151	-.0360	-.0230	-.2366
$M_o = 1.30$							
0.2541	-0.0008	-0.0561	-0.5982	0.2487	0.0244	0.0003	0.2200
.2858	-.0005	-.0505	-.6508	.2255	.0233	.0016	.2440
.3173	.0001	-.0364	-.6987	.2041	.0183	.0014	.2644
.3488	.0009	-.0117	-.7430	.1845	.0109	.0003	.2742
.3800	.0021	.0258	-.7856	.1667	.0019	-.0016	.2700
.4111	.0037	.0790	-.8300	.1507	-.0099	-.0048	.2285
.4418	.0057	.1508	-.8834	.1366	-.0236	-.0090	.1184
.4721	.0083	.2449	-.9608	.1242	-.0404	-.0169	-.0818
.5020	.0114	.3652	-1.0977	.1136	-.0628	-.0294	-.4943
.5072	.0120	.3893	-1.1334	.1119	-.0671	-.0324	-.5971
.5416	.0167	.5777	-1.3074	.1020	-.0834	-.0549	-1.5080

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE II

AUXILIARY FUNCTIONS FOR CALCULATING THE STABILITY OF THE

LAMINAR BOUNDARY LAYER FOR NONINSULATED SURFACE

c	λ	v	L	H_1	H_2	M_3	N_3
$M_0 = 0.70;\ T_1 = 0.70$							
0.0262	0.0056	0.0825	0.0635	0.6102	0.3272	0.0524	0.2748
.0521	.0112	.1645	.0949	.5725	.3157	.0502	.2920
.0777	.0166	.2466	.1184	.5367	.3045	.0481	.3081
.1030	.0220	.3297	.1400	.5026	.2936	.0458	.3233
.1281	.0274	.4146	.1632	.4703	.2828	.0433	.3380
.1529	.0327	.5023	.1904	.4396	.2724	.0412	.3519
.1701	.0365	.5661	.2130	.4191	.2651	.0395	.3610
.1726	.0370	.5754	.2163	.4162	.2642	.0394	.3623
$M_0 = 0.70;\ T_1 = 0.80$							
0.0237	0.0033	0.0486	0.0279	0.5954	0.2811	0.0493	0.1369
.0472	.0066	.0965	.0374	.5620	.2737	.0475	.1504
.0705	.0099	.1443	.0430	.5300	.2663	.0457	.1635
.0937	.0132	.1925	.0482	.4994	.2590	.0437	.1763
.1168	.0164	.2417	.0550	.4701	.2514	.0417	.1882
.1397	.0197	.2926	.0649	.4420	.2439	.0397	.2001
.1625	.0230	.3457	.0789	.4152	.2363	.0378	.2110
.1851	.0263	.4017	.0982	.3897	.2287	.0359	.2213
.2075	.0297	.4614	.1236	.3654	.2210	.0339	.2311
.2298	.0331	.5253	.1562	.3424	.2133	.0321	.2400
.2409	.0349	.5592	.1754	.3313	.2094	.0310	.2443
.2475	.0359	.5801	.1877	.3248	.2071	.0303	.2465
$M_0 = 0.70;\ T_1 = 0.90$							
0.0433	0.0036	0.0517	0.0051	0.5506	0.2410	0.0435	0.1426
.0863	.0072	.1028	-.0047	.4939	.2304	.0404	.1638
.1291	.0108	.1568	-.0111	.4414	.2191	.0370	.1846
.1714	.0145	.2173	-.0079	.3930	.2074	.0337	.2032
.2135	.0185	.2885	.0096	.3485	.1951	.0304	.2203
.2551	.0227	.3746	.0462	.3080	.1825	.0272	.2339
.2963	.0274	.4805	.1073	.2715	.1698	.0240	.2462
.3166	.0299	.5426	.1489	.2547	.1637	.0224	.2517
.3268	.0312	.5762	.1776	.2466	.1606	.0217	.2541
$M_0 = 0.70;\ T_1 = 1.25$							
0.0346	-0.0016	-0.0237	-0.0476	0.5100	0.1750	0.0324	0.1462
.0692	-.0032	-.0476	-.0797	.4678	.1710	.0310	.1634
.1040	-.0048	-.0698	-.1013	.4276	.1661	.0292	.1794
.1389	-.0062	-.0886	-.1132	.3896	.1600	.0272	.1956
.1738	-.0076	-.1021	-.1155	.3538	.1529	.0251	.2108
.2088	-.0087	-.1085	-.1081	.3202	.1448	.0228	.2238
.2439	-.0095	-.1057	-.0912	.2888	.1354	.0208	.2342
.2789	-.0101	-.0917	-.0645	.2597	.1249	.0185	.2402
.3138	-.0103	-.0641	-.0281	.2330	.1133	.0161	.2409
.3485	-.0100	-.0203	.0179	.2086	.1008	.0139	.2297
.3831	-.0092	.0427	.0734	.1865	.0870	.0113	.2069
.4174	-.0079	.1286	.1373	.1668	.0728	.0083	.1616
.4512	-.0059	.2414	.2071	.1495	.0582	.0042	.0816
.4846	-.0031	.3859	.2770	.1345	.0427	-.0012	-.0601
.5092	-.0006	.5184	.3212	.1248	.0314	-.0067	-.2262
.5190	.0006	.5779	.3349	.1212	.0269	-.0091	-.3028

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE III

PHASE VELOCITY, WAVE NUMBER, AND REYNOLDS NUMBER FOR NEUTRAL SUBSONIC DISTURBANCE (STABILITY LIMITS) FOR INSULATED SURFACE

c	α	R	α_θ	R_θ
$M_0 = 0$				
0.0372	0.0321	25,500,000	0.0038	3,030,000
.0744	.0685	1,500,000	.0082	178,000
.1115	.1103	278,000	.0131	33,100
.1486	.1585	83,000	.0189	9,880
.1857	.2146	32,600	.0255	3,880
.2226	.2808	14,800	.0334	1,760
.2594	.3590	7,700	.0427	917
.2960	.4535	4,420	.0540	526
.3323	.5707	2,760	.0679	329
.3682	.7243	1,850	.0862	220
.4037	.9589	1,360	.1142	162
.4143	1.0770	1,280	.1282	153
.4143	1.2730	1,530	.1515	182
.4037	1.2940	1,880	.1540	223
.3682	1.1960	3,530	.1424	421
.3323	1.0400	6,710	.1238	799
.2960	.8728	13,300	.1039	1,580
.2594	.7177	27,500	.0854	3,270
$M_0 = 0.50$				
0.0362	0.0251	36,600,000	0.0029	4,270,000
.0723	.0538	2,130,000	.0063	248,000
.1085	.0868	392,000	.0101	45,700
.1446	.1250	116,000	.0146	13,500
.1806	.1695	44,500	.0198	5,190
.2166	.2216	20,200	.0258	2,360
.2525	.2829	10,400	.0330	1,210
.2882	.3556	5,850	.0414	682
.3237	.4442	3,570	.0518	416
.3588	.5549	2,330	.0647	272
.3936	.6993	1,620	.0815	189
.4280	.9301	1,230	.1084	144
.4306	.9558	1,220	.1114	142
.4362	1.0140	1,190	.1182	139
.4362	1.1880	1,410	.1384	164
.4306	1.2150	1,580	.1416	184
.4280	1.2150	1,660	.1416	194
.3936	1.1240	3,080	.1310	359
.3588	.9788	5,670	.1141	661
.3237	.8272	10,800	.0964	1,260
.2882	.6869	21,100	.0800	2,460
$M_0 = 0.70$				
0.0353	0.0191	53,400,000	0.0022	6,100,000
.0705	.0415	3,060,000	.0047	349,000
.1058	.0677	555,000	.0077	63,400
.1410	.0984	161,000	.0112	18,400
.1762	.1344	61,100	.0154	6,980
.2114	.1766	27,300	.0202	3,120
.2464	.2268	13,800	.0259	1,580
.2813	.2857	7,630	.0326	872
.3161	.3570	4,550	.0408	520
.3505	.4433	2,900	.0506	331
.3847	.5515	1,960	.0630	224
.4185	.6951	1,420	.0794	162
.4352	.7917	1,230	.0904	141
.4452	.8655	1,160	.0989	132
.4559	.9704	1,110	.1108	127
.4559	1.1230	1,330	.1283	152
.4452	1.1420	1,650	.1304	189
.4352	1.1230	1,980	.1283	227
.4185	1.0720	2,670	.1225	305
.3847	.9381	4,810	.1072	550
.3505	.7965	8,880	.0910	1,010
.3161	.6659	16,700	.0761	1,910

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE III - Concluded

PHASE VELOCITY, WAVE NUMBER, AND REYNOLDS NUMBER FOR NEUTRAL SUBSONIC DISTURBANCE (STABILITY LIMITS) FOR INSULATED SURFACE - Concluded

c	α	R	α_θ	R_θ
		M_o = 0.90		
0.0334	0.0107	111,000,000	0.0012	12,600,000
.0667	.0248	5,960,000	.0028	679,000
.1001	.0421	1,030,000	.0048	117,000
.1335	.0632	290,000	.0072	33,000
.1669	.0885	106,000	.0101	12,100
.2002	.1186	46,000	.0135	5,240
.2335	.1540	22,600	.0175	2,570
.2666	.1961	12,100	.0223	1,380
.2997	.2459	7,020	.0280	800
.3326	.3053	4,320	.0348	492
.3652	.3777	2,820	.0430	321
.3976	.4638	1,950	.0529	222
.4296	.5733	1,410	.0653	161
.4612	.7260	1,090	.0827	125
.4636	.7396	1,080	.0843	123
.4812	.8810	1,010	.1004	115
.4812	1.0130	1,230	.1154	140
.4636	1.0120	1,740	.1153	199
.4612	1.0070	1,820	.1148	207
.4296	.9027	3,180	.1029	363
.3976	.7823	5,590	.0892	637
.3652	.6642	9,940	.0757	1,130
.3326	.5523	18,500	.0629	2,100
		M_o = 1.10		
0.0990	0.0086	5,730,000	0.0009	618,000
.1320	.0268	769,000	.0029	82,900
.1650	.0468	224,000	.0050	24,100
.1980	.0707	85,000	.0076	9,160
.2309	.0991	38,300	.0107	4,130
.2638	.1329	19,300	.0143	2,080
.2965	.1727	10,600	.0186	1,140
.3292	.2200	6,260	.0237	675
.3616	.2755	3,920	.0297	423
.3938	.3417	2,610	.0368	281
.4246	.4159	1,850	.0448	199
.4572	.5193	1,350	.0560	146
.4836	.6268	1,100	.0676	119
.5104	.8010	991	.0864	107
.5104	.9165	1,220	.0988	131
.4836	.8927	2,060	.0962	223
.4572	.8023	3,320	.0865	358
.4246	.6785	5,930	.0732	639
.3938	.5766	10,400	.0622	1,120
		M_o = 1.30		
0.2541	0.0451	63,800	0.0047	6,630
.2858	.0818	24,800	.0085	2,570
.3173	.1202	12,300	.0125	1,280
.3488	.1636	6,990	.0170	726
.3800	.2132	4,280	.0222	445
.4111	.2707	2,800	.0281	291
.4418	.3377	1,930	.0351	201
.4721	.4166	1,420	.0433	147
.5020	.5123	1,110	.0532	115
.5072	.5316	1,070	.0552	111
.5416	.7582	886	.0788	92
.5416	.8931	1,080	.0928	112
.5072	.7781	2,310	.0809	241
.5020	.7592	2,550	.0789	265
.4721	.6458	4,500	.0671	468
.4418	.5401	7,980	.0561	829

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE IV

PHASE VELOCITY, WAVE NUMBER, AND REYNOLDS NUMBER FOR NEUTRAL SUBSONIC DISTURBANCE (STABILITY LIMITS) FOR NONINSULATED SURFACES

c	α	R	α_θ	R_θ
$M_o = 0.70$; $T_1 = 0.70$				
0.0262	0.0339	82,400,000	0.0041	9,900,000
.0521	.0734	5,360,000	.0088	644,000
.0777	.1188	1,110,000	.0143	133,000
.1030	.1708	371,000	.0205	44,600
.1281	.2308	161,000	.0277	19,300
.1529	.3030	83,400	.0364	10,000
.1701	.3670	57,200	.0441	6,870
.1726	.3777	54,400	.0454	6,540
.1726	.4986	69,000	.0599	8,280
.1701	.4977	73,900	.0598	8,870
.1529	.4732	121,000	.0568	14,500
.1281	.4175	270,000	.0502	32,400
.1030	.3460	711,000	.0416	85,400
.0777	.2620	2,500,000	.0315	300,000
.0521	.1713	14,600,000	.0206	1,750,000
$M_o = 0.70$; $T_1 = 0.80$				
0.0237	0.0237	157,000,000	0.0028	18,300,000
.0472	.0504	9,910,000	.0059	1,150,000
.0705	.0804	1,970,000	.0094	230,000
.0937	.1138	633,000	.0133	73,700
.1168	.1509	263,000	.0176	30,600
.1397	.1923	129,000	.0224	15,000
.1625	.2382	70,900	.0278	8,260
.1851	.2908	42,600	.0339	4,960
.2075	.3510	27,500	.0409	3,200
.2298	.4237	18,800	.0494	2,190
.2409	.4668	15,900	.0544	1,860
.2475	.4962	14,500	.0578	1,690
.2475	.6308	18,500	.0735	2,160
.2409	.6233	21,400	.0726	2,500
.2298	.6056	27,200	.0706	3,170
.2075	.5609	44,900	0654	5,230
.1851	.5062	77,400	.0590	9,010
.1625	.4465	141,000	.0520	16,400
.1397	.3827	280,000	.0446	32,600
.1168	.3164	630,000	.0369	73,400
.0937	.2489	1,690,000	.0290	197,000
.0705	.1822	5,890,000	.0212	686,000

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE IV - Concluded

PHASE VELOCITY, WAVE NUMBER, AND REYNOLDS NUMBER FOR

NEUTRAL SUBSONIC DISTURBANCE (STABILITY LIMITS) FOR

NONINSULATED SURFACES - Concluded

c	α	R	α_θ	R_θ
$M_o = 0.70$; $T_1 = 0.90$				
0.0433	0.0368	17,100,000	0.0042	1,930,000
.0863	.0815	1,040,000	.0092	118,000
.1291	.1353	200,000	.0153	22,700
.1714	.1996	62,500	.0226	7.070
.2135	.2775	25,500	.0314	2,880
.2551	.3728	12,400	.0422	1,410
.2963	.4980	6,970	.0563	789
.3166	.5814	5,520	.0658	624
.3268	.6347	4,990	.0718	565
.3268	.7817	6,500	.0884	735
.3166	.7701	7,920	.0871	895
.2963	.7307	11,600	.0827	1,310
.2551	.6275	25,200	.0710	2,850
.2135	.5133	60,300	.0581	6,820
.1714	.3972	170,000	.0449	19,200
.1291	.2858	617,000	.0323	69,800
.0863	.1793	3,740,000	.0203	423,000
$M_o = 0.70$; $T_1 = 1.25$				
0.0346	0.0160	78,800,000	0.0016	8,090,000
.0692	.0346	4,380,000	.0036	450,000
.1040	.0564	770,000	.0058	79,000
.1389	.0819	217,000	.0084	22,200
.1738	.1120	78,900	.0115	8,100
.2088	.1477	34,000	.0152	3,490
.2439	.1899	16,500	.0195	1,700
.2789	.2403	8,830	.0247	907
.3138	.3002	5,070	.0308	520
.3485	.3722	3,110	.0382	319
.3831	.4594	2,020	.0471	207
.4174	.5668	1,380	.0582	142
.4512	.7061	1,000	.0725	103
.4846	.9067	760	.0931	78
.5092	1.1800	643	.1211	66
.5190	1.4480	615	.1486	63
.5190	1.5880	640	.1630	66
.5092	1.7250	806	.1770	83
.4846	1.5370	1,390	.1577	142
.4512	1.2580	2,740	.1291	281
.4174	1.0330	5,360	.1060	550

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

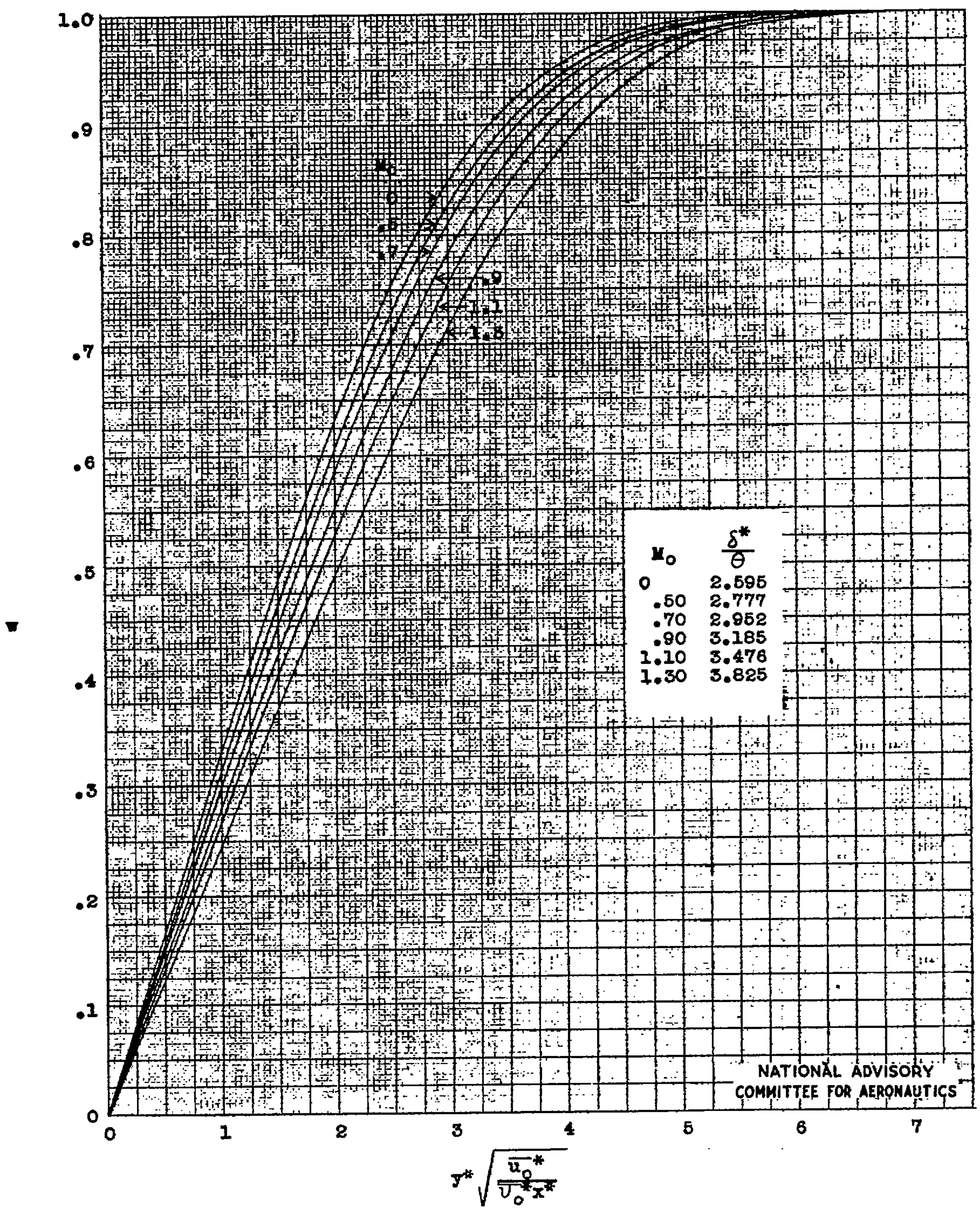
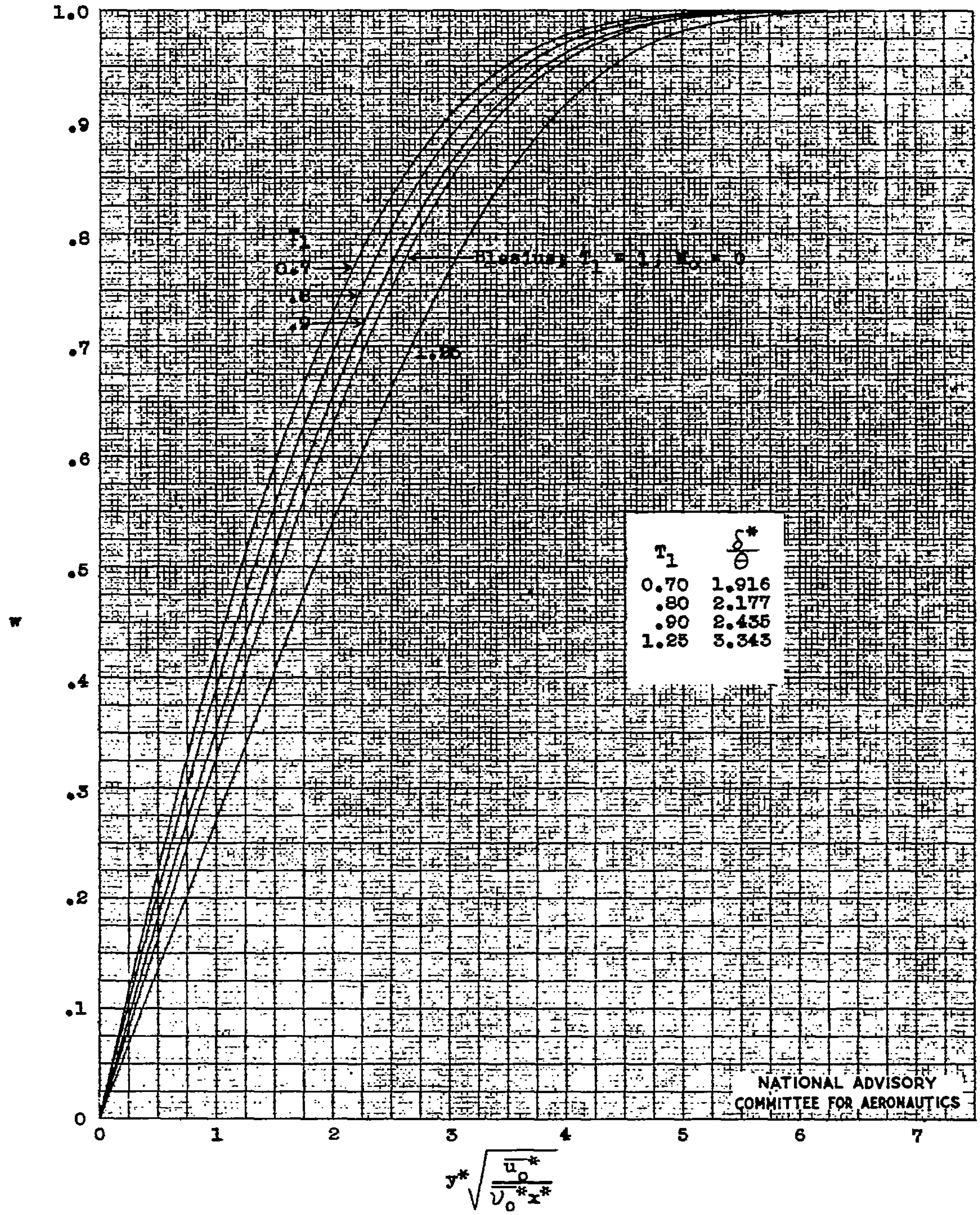
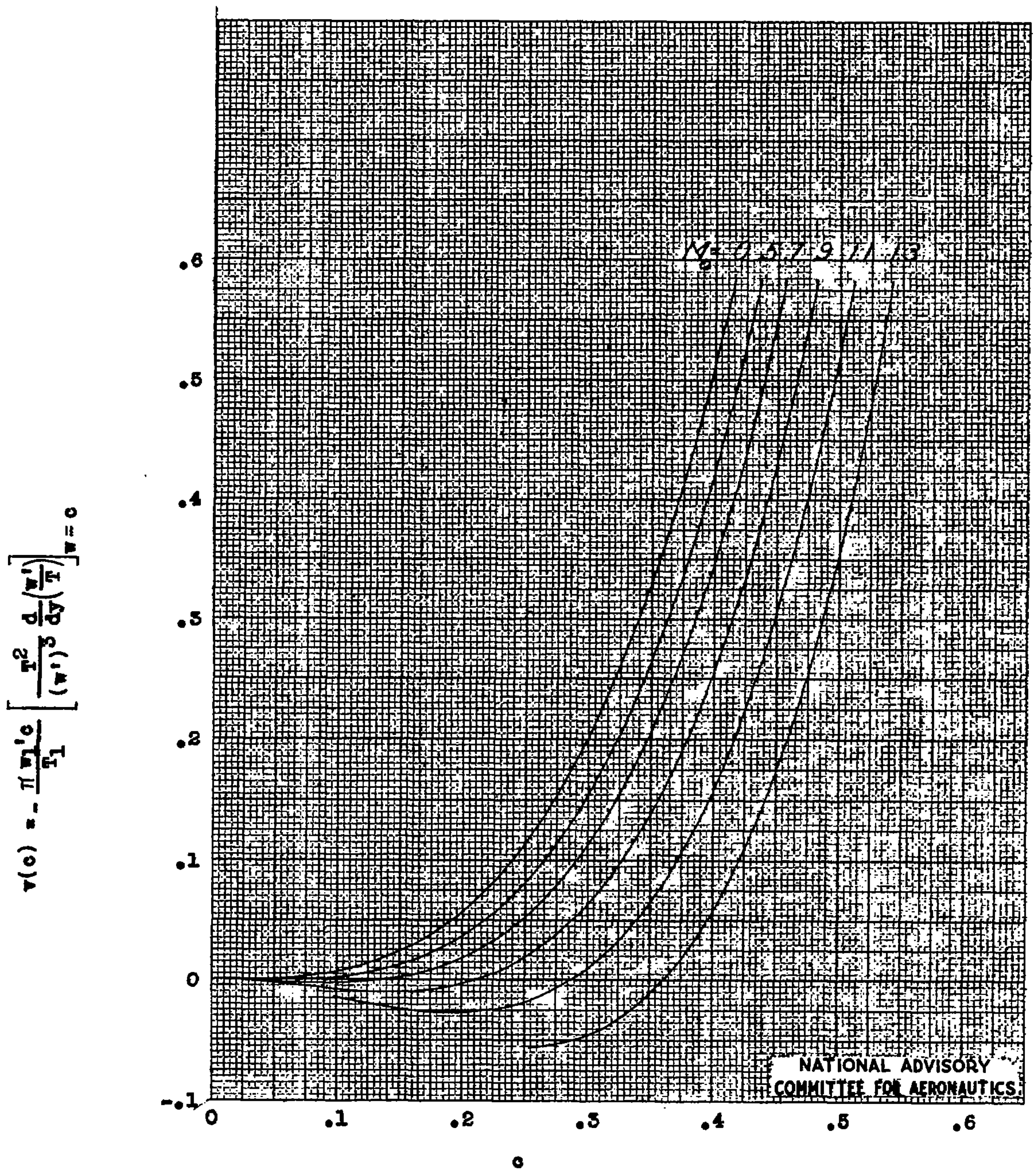


Figure 1.- Boundary-layer velocity profiles for insulated surface. Since σ is taken equal to unity, the temperature profile is given by $T = T_1 - \left[T_1 - \left(1 + \frac{\gamma - 1}{2} M_o^2\right)\right] w - \frac{\gamma - 1}{2} M_o^2 w^2$.



T_1	$\frac{\delta^*}{\theta}$
0.70	1.916
.80	2.177
.90	2.435
1.25	3.343

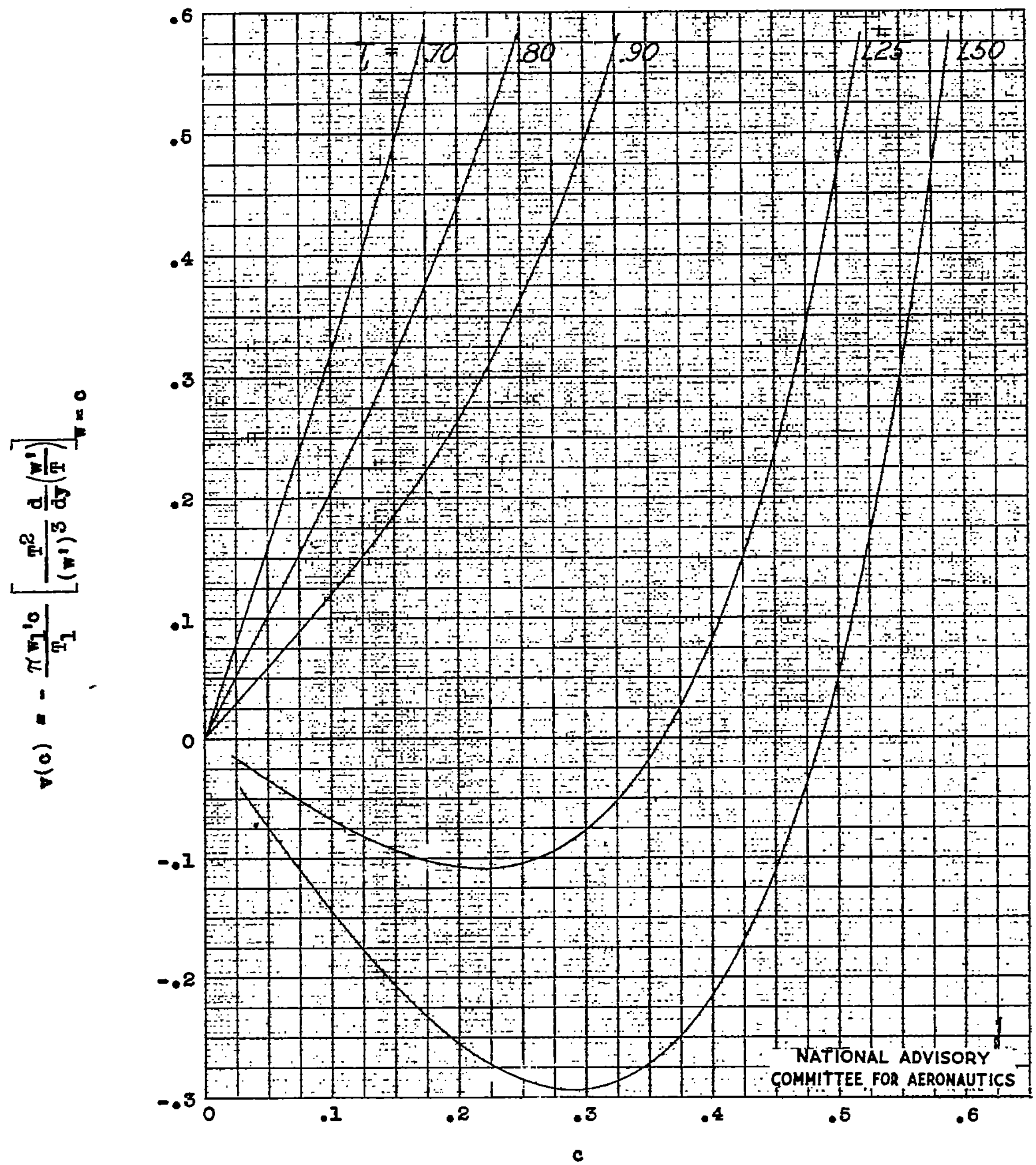
Figure 2.- Boundary-layer velocity profiles for noninsulated surface. M_o = 0.70. T_1 is the ratio of surface temperature (deg abs.) to free-stream temperature (deg abs.). Inflection more pronounced and farther out into fluid for T_1 = 1.25 than for insulated surface (T_1 = 1.10). No inflection for T_1 = 0.70, 0.80, 0.90.



(a) Insulated surface.

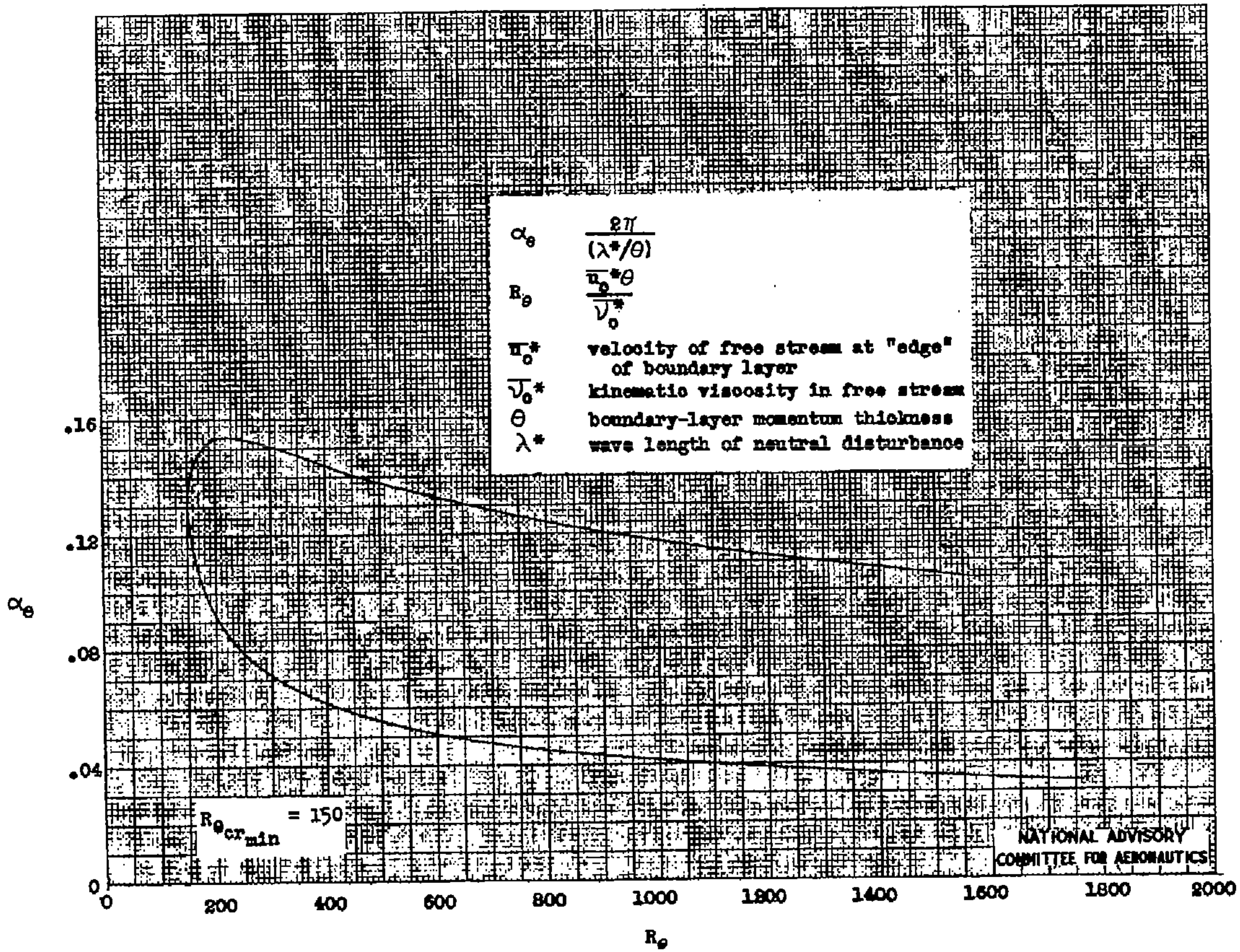
Figure 3.- Distribution of $\frac{d}{dy}\left(\rho \frac{dw}{dy}\right)$ across boundary layer, as expressed by the function $v(c)$ which appears directly in stability calculation. The value of $c = c_0$ at which $(1 - 2\lambda)v = 0.580$ (λ is small) is a measure of the stability of a given laminar boundary-layer flow.

$$R_{\theta_{cr_{min}}} \approx \frac{6}{T_1} \frac{\left[T(c_0)\right]^{1.76}}{c_0{}^4 \sqrt{1 - M_0^2(1 - c_0)^2}} .$$



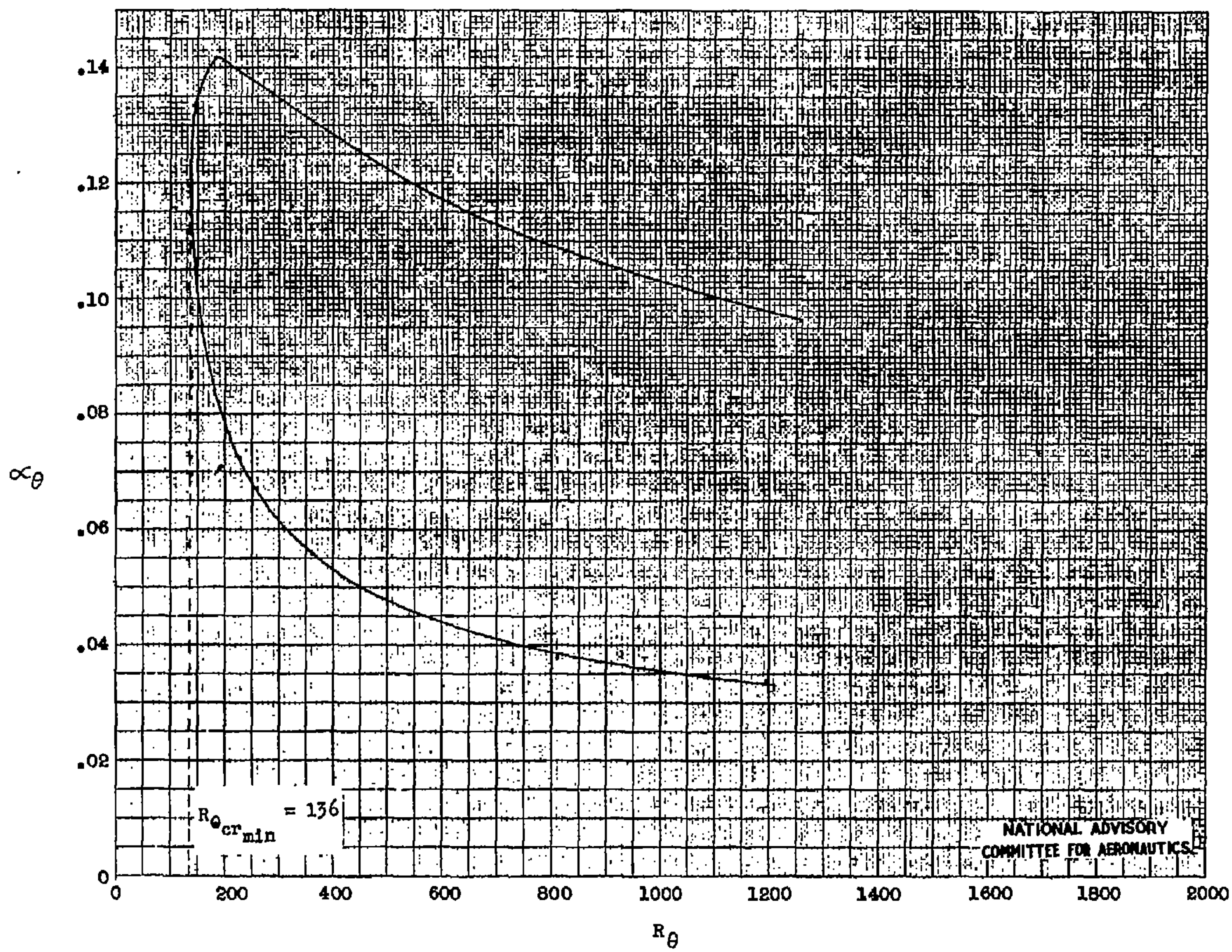
(b) Noninsulated surface. $M_o = 0.70$.

Figure 3.- Concluded.



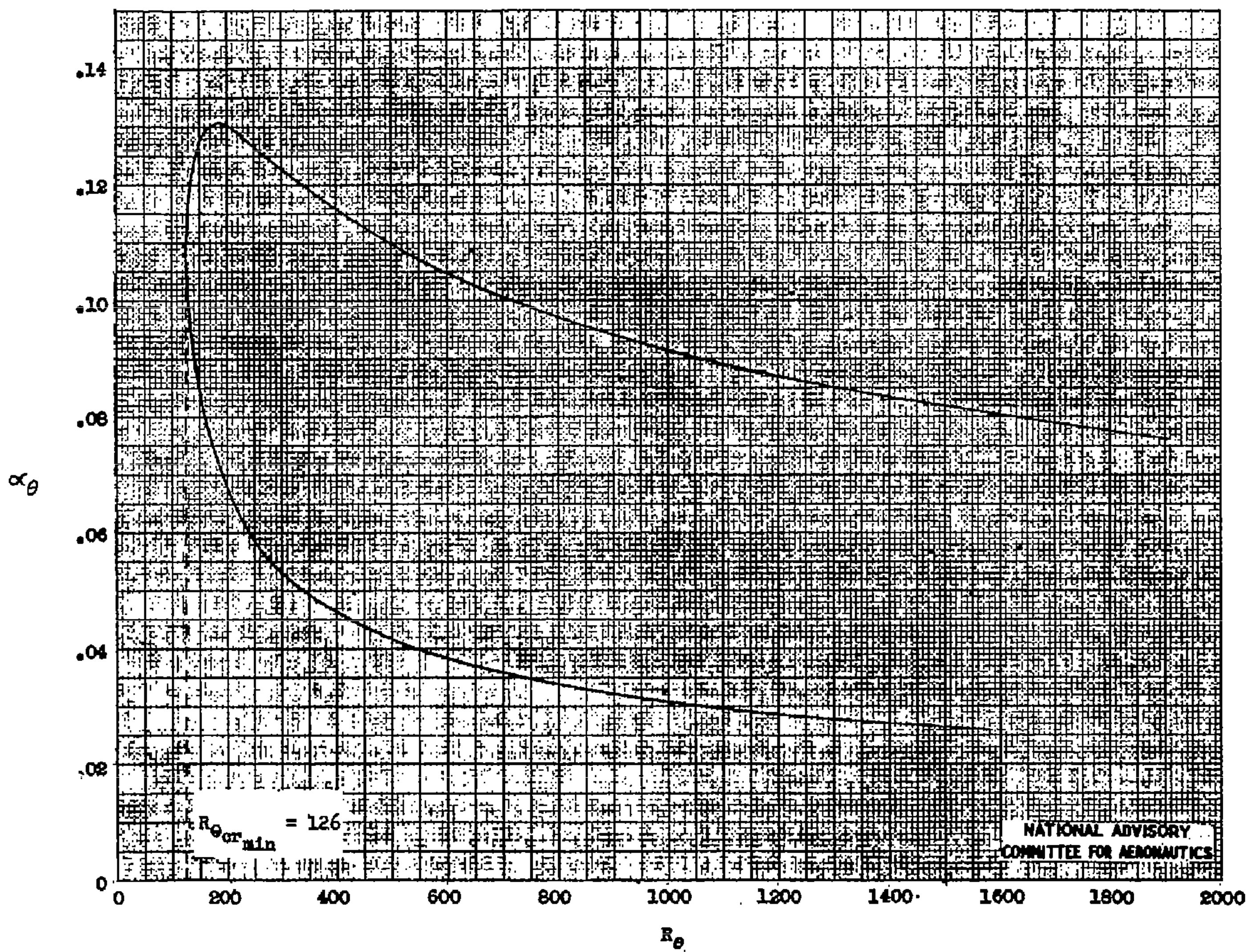
(a) Insulated surface. $M_o = 0$ (Blasius flow).

Figure 4.- Wave number α_θ against Reynolds number R_θ for neutral stability of laminar boundary-layer flow.



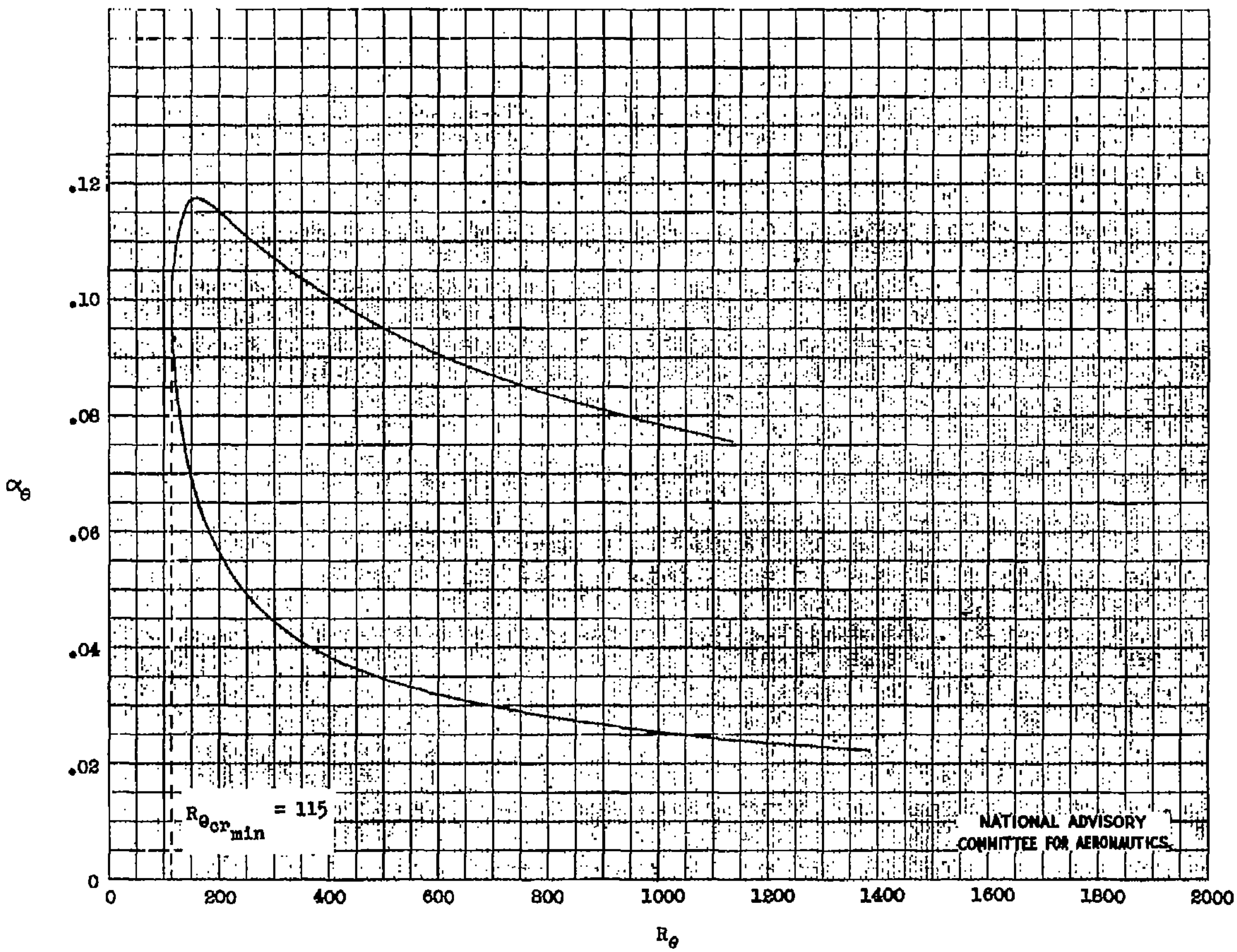
(b) Insulated surface. $M_0 = 0.50$.

Figure 4.- Continued.



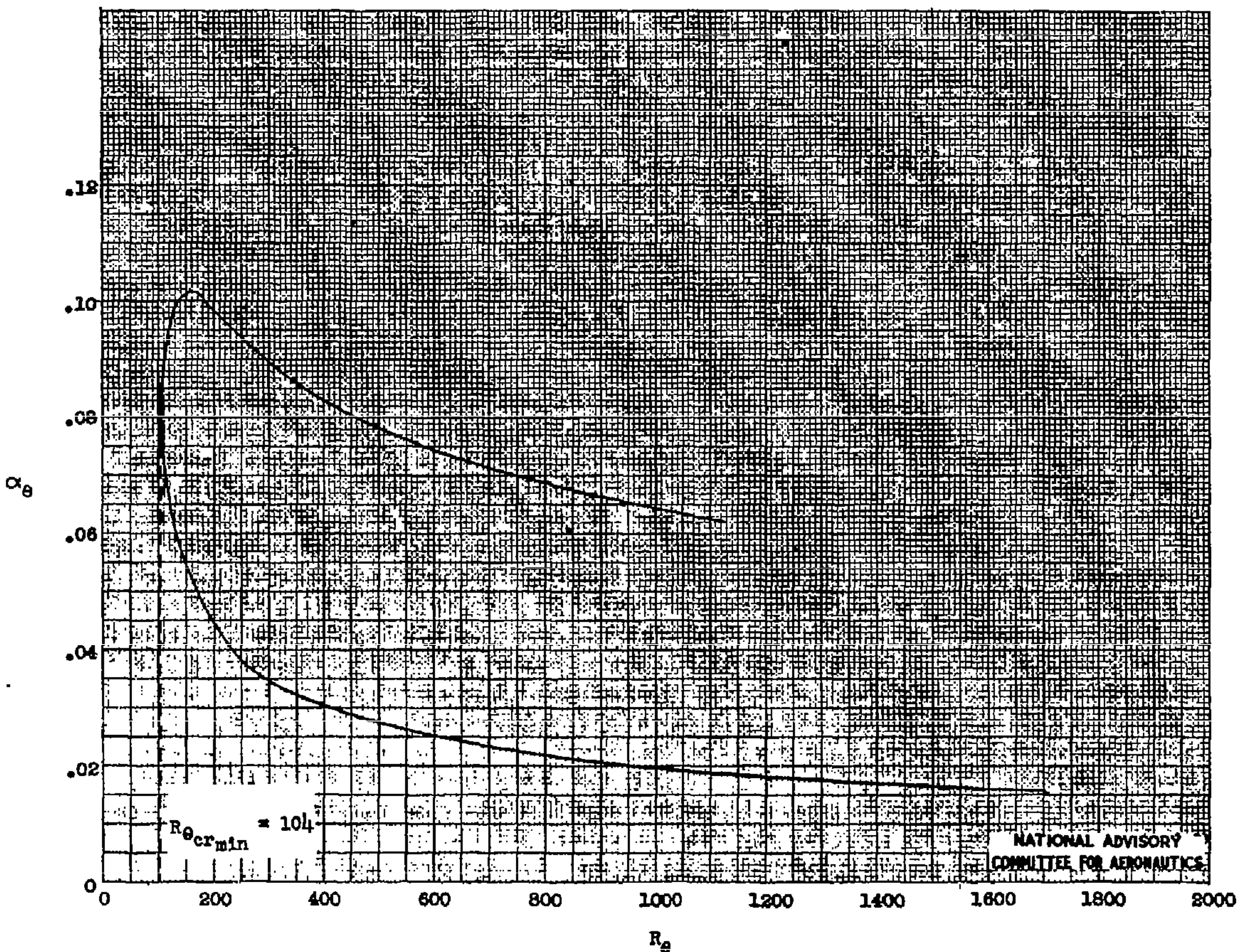
(c) Insulated surface. $M_0 = 0.70$.

Figure 4.- Continued.



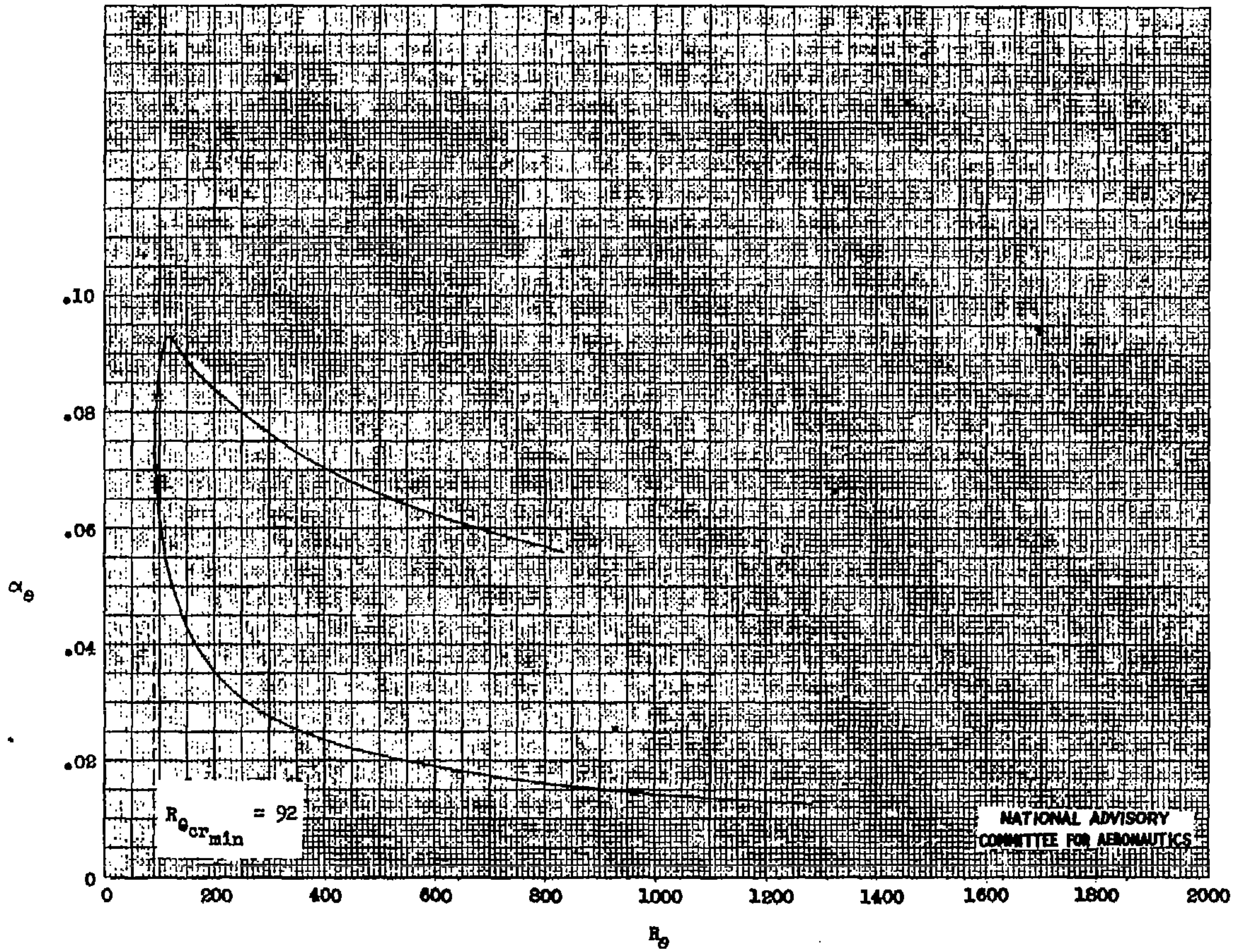
(d) Insulated surface. $M_0 = 0.90$.

Figure 4.- Continued.



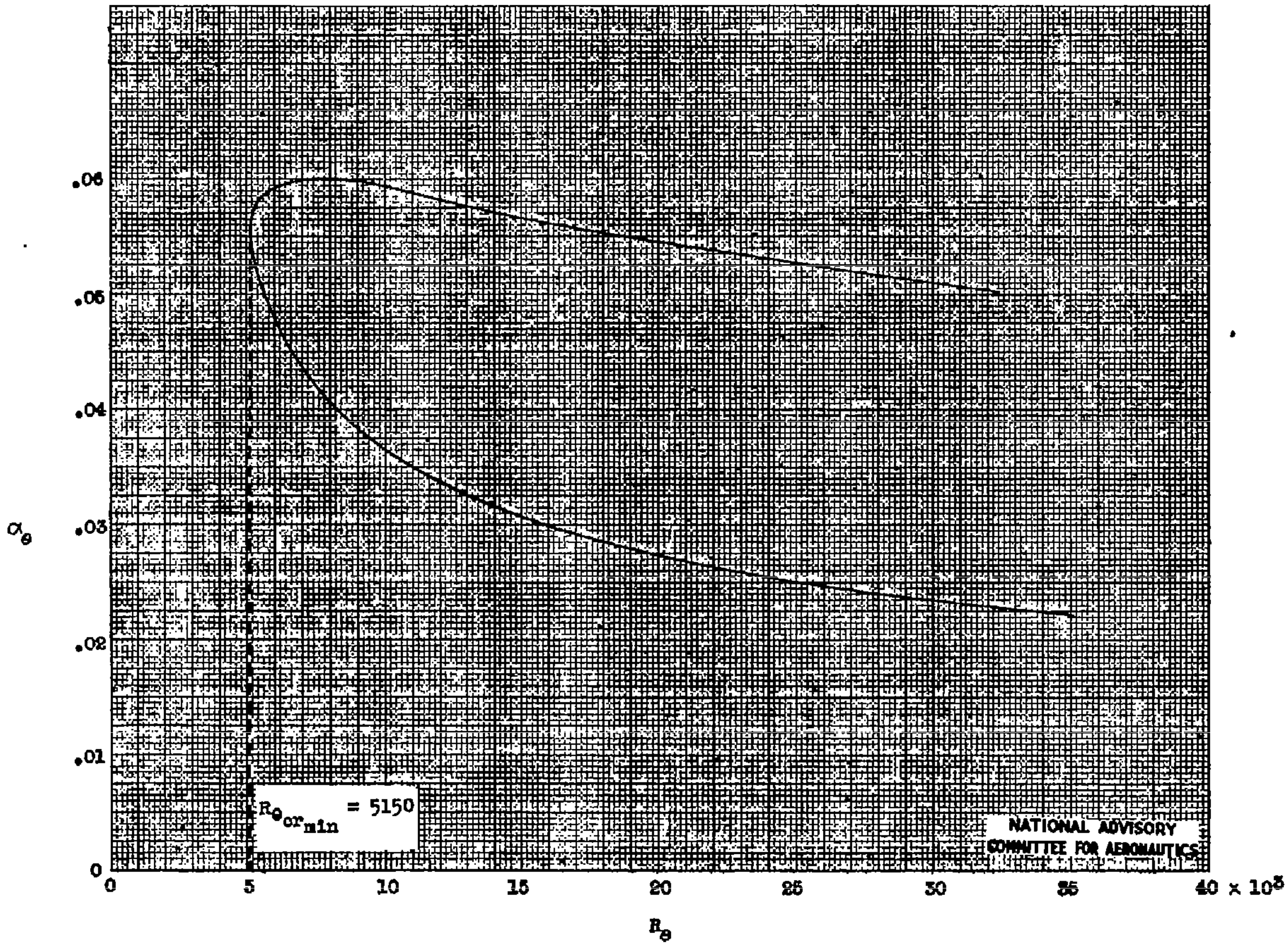
(e) Insulated surface. $M_0 = 1.10$.

Figure 4.- Continued.



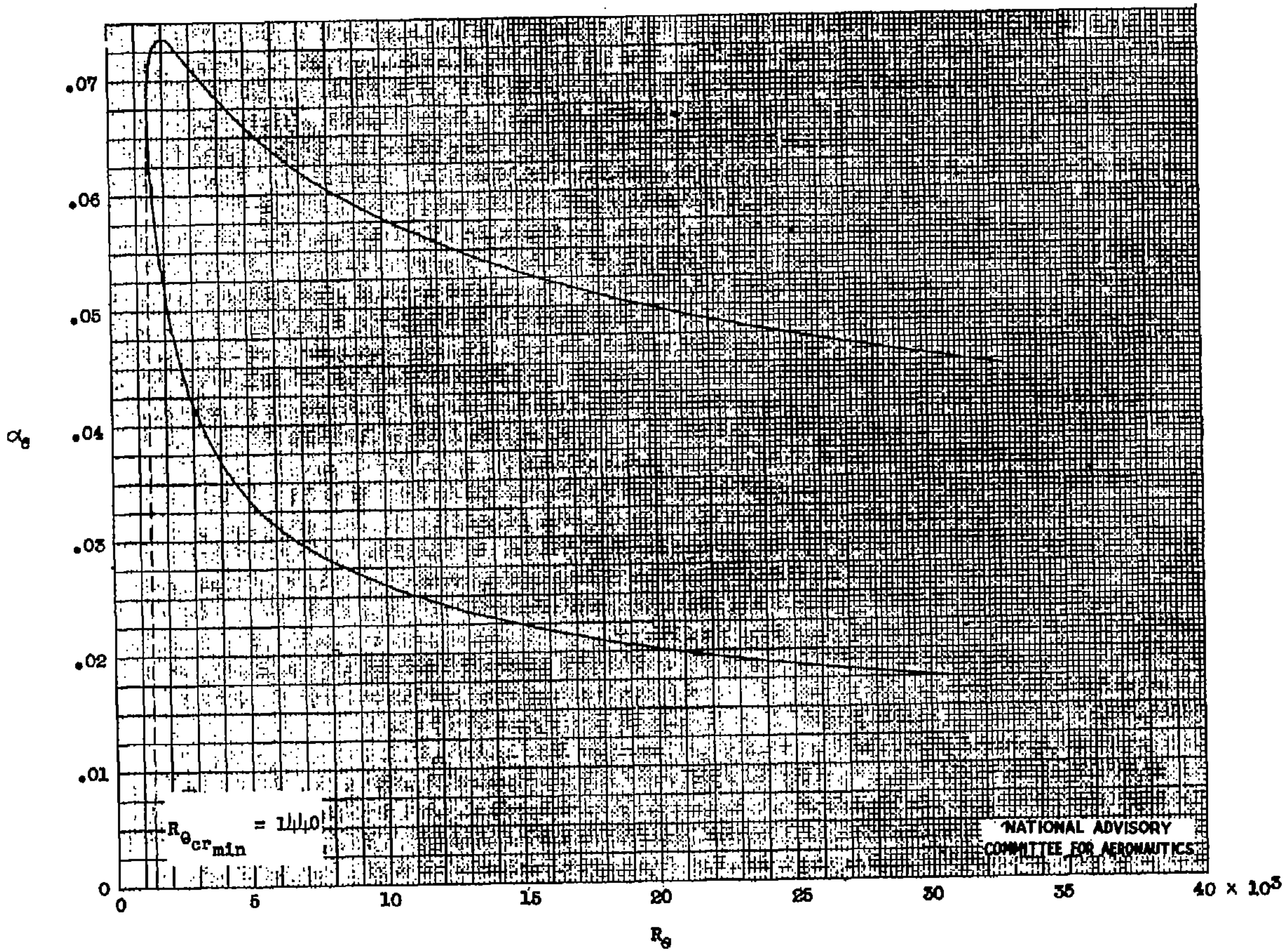
(f) Insulated surface. $M_0 = 1.30$.

Figure 4.- Continued.



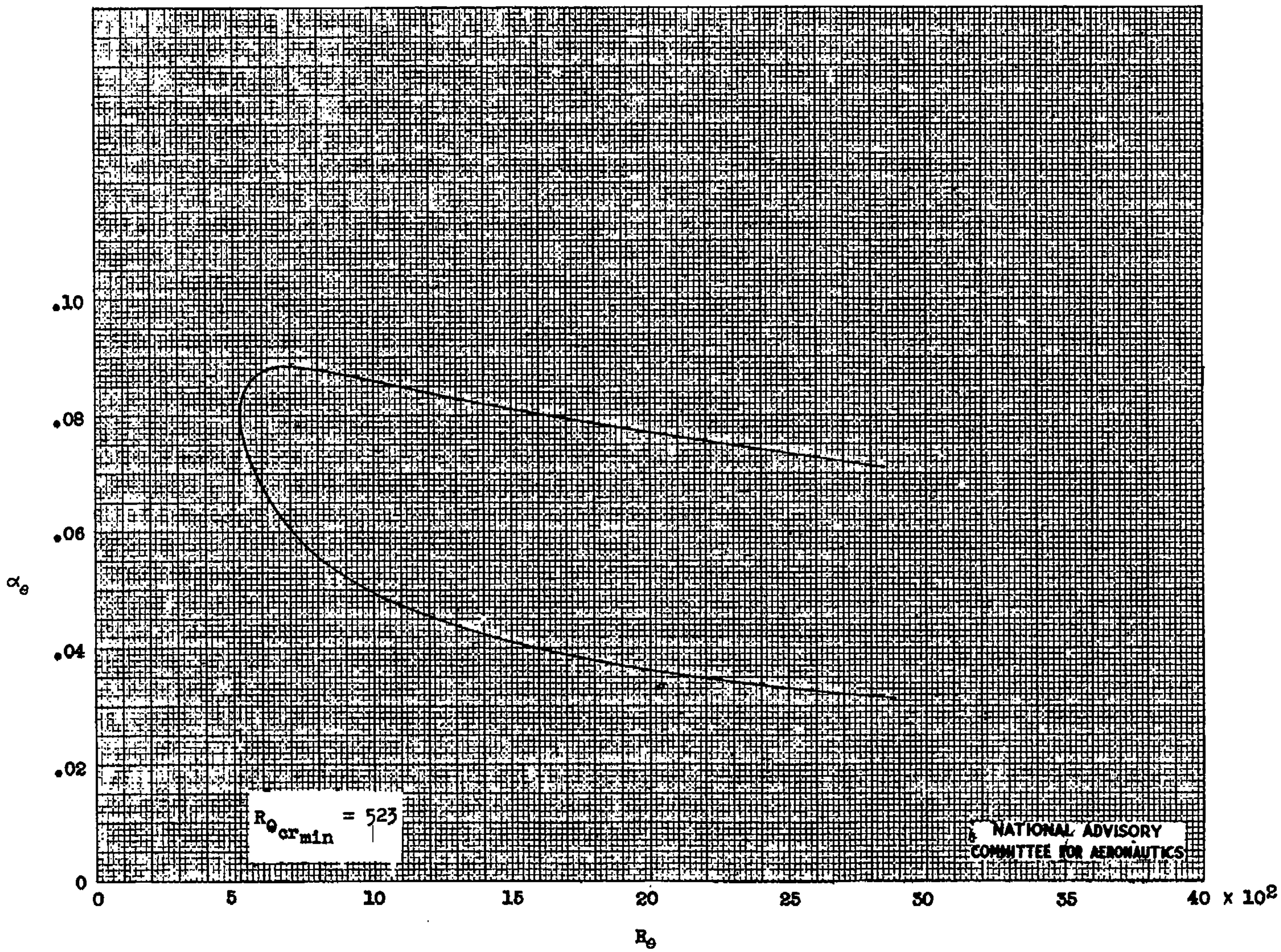
(g) Noninsulated surface. $M_0 = 0.70$; $T_1 = 0.70$; $\left(\frac{\partial T}{\partial y}\right)_{y=0} > 0$.

Figure 4.- Continued.



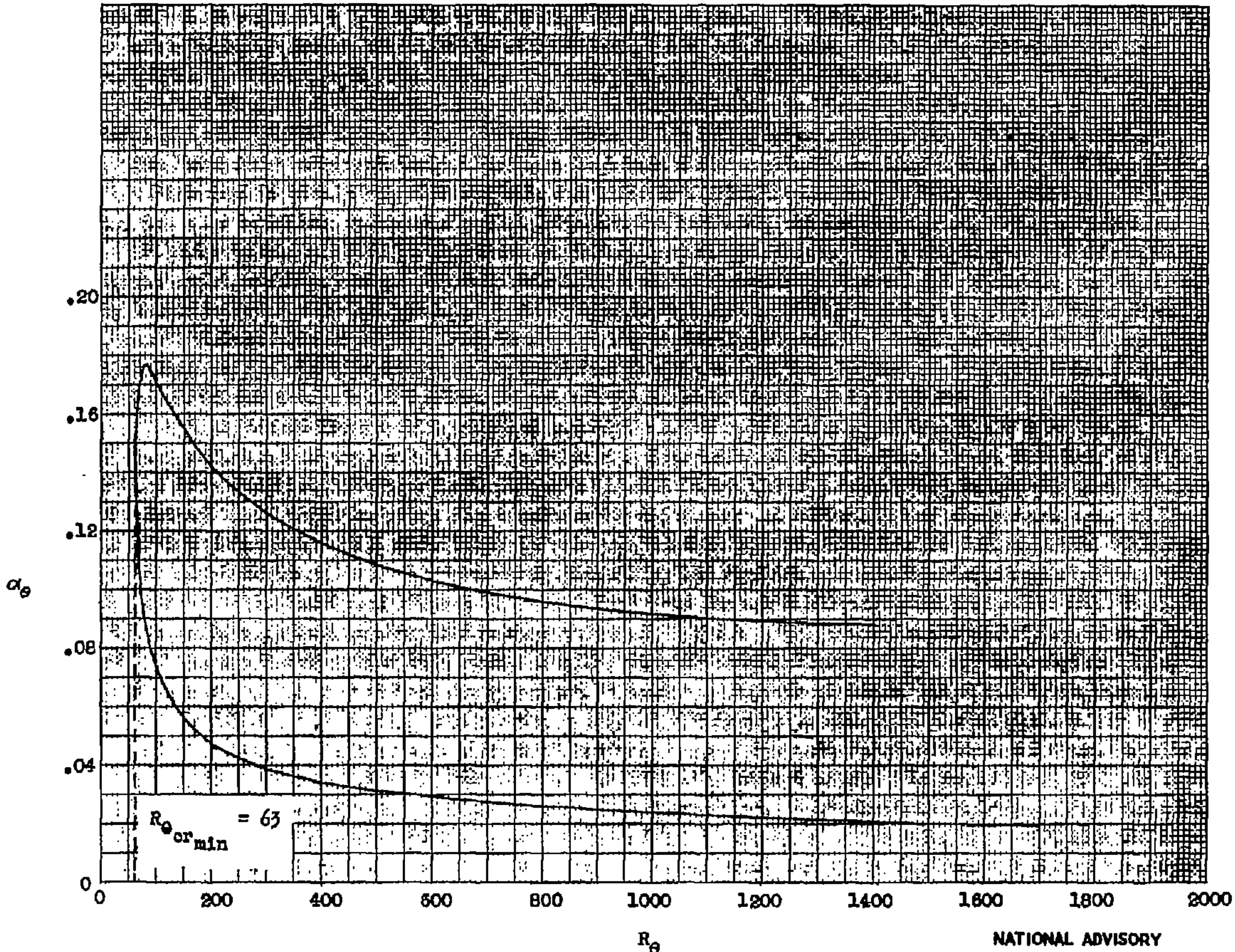
(h) Noninsulated surface. $M_0 = 0.70$; $T_1 = 0.80$; $\left(\frac{\partial T}{\partial y}\right)_{y=0} > 0$.

Figure 4.- Continued.



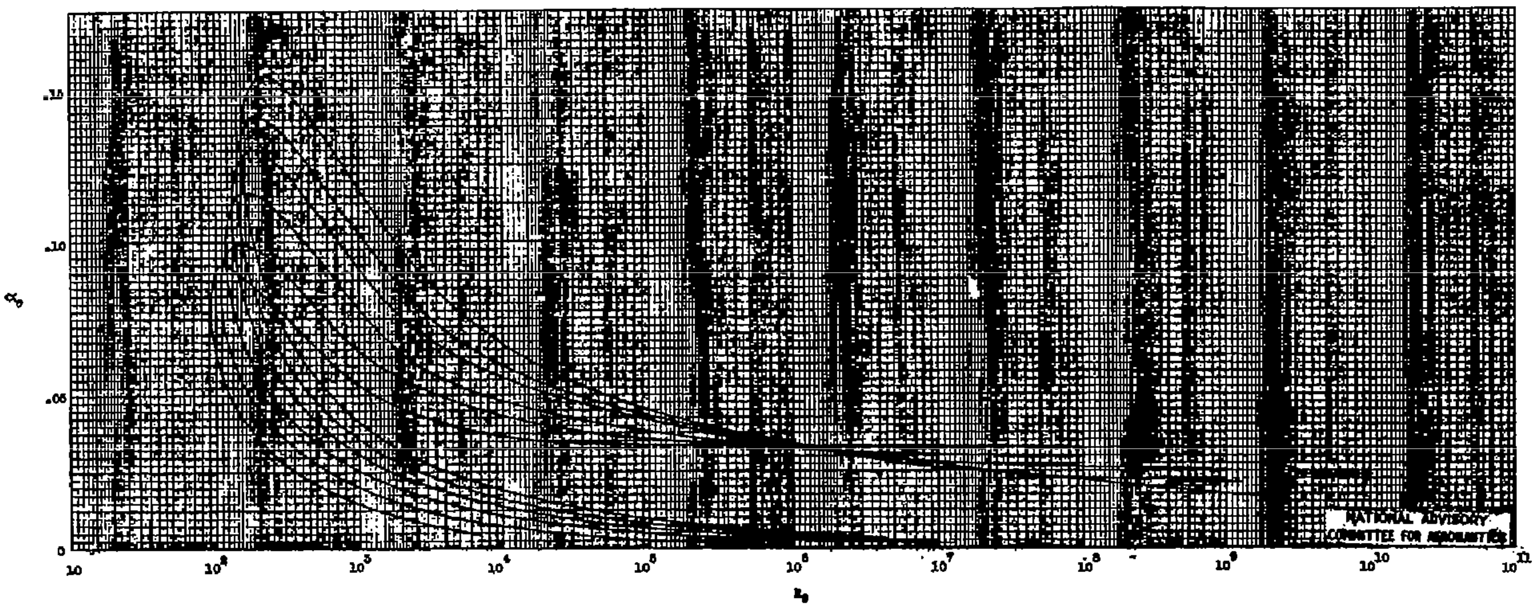
(i) Noninsulated surface. $M_0 = 0.70$; $\tau_1 = 0.90$; $\left(\frac{\partial T}{\partial y}\right)_{y=0} > 0$.

Figure 4.- Continued.



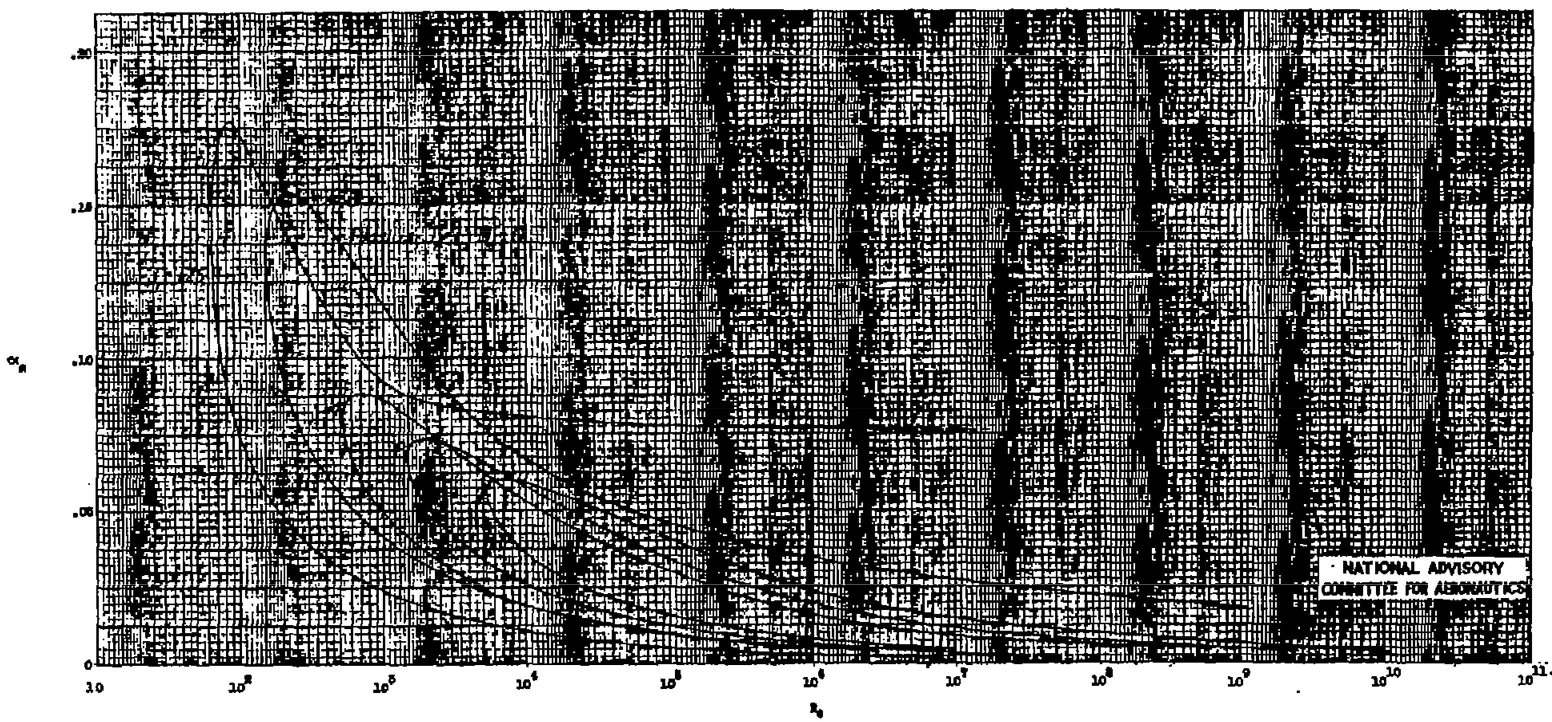
(j) Noninsulated surface. $M_0 = 0.70$; $T_1 = 1.25$; $\left(\frac{\partial T}{\partial y}\right)_{y=0} < 0$.

Figure 4.- Continued.



(k) Insulated surface. Note that $\alpha \rightarrow \alpha_S \neq 0$ as $R \rightarrow \infty$ for insulated surface. ($M_0 \neq 0$.)

Figure 4.- Continued.



(*l*) Noninsulated surface. $M_0 = 0.70$. Note that $\alpha \rightarrow \alpha_s \neq 0$ as $R \rightarrow \infty$ when $T_1 = 1.25$.

Figure 4.- Concluded.

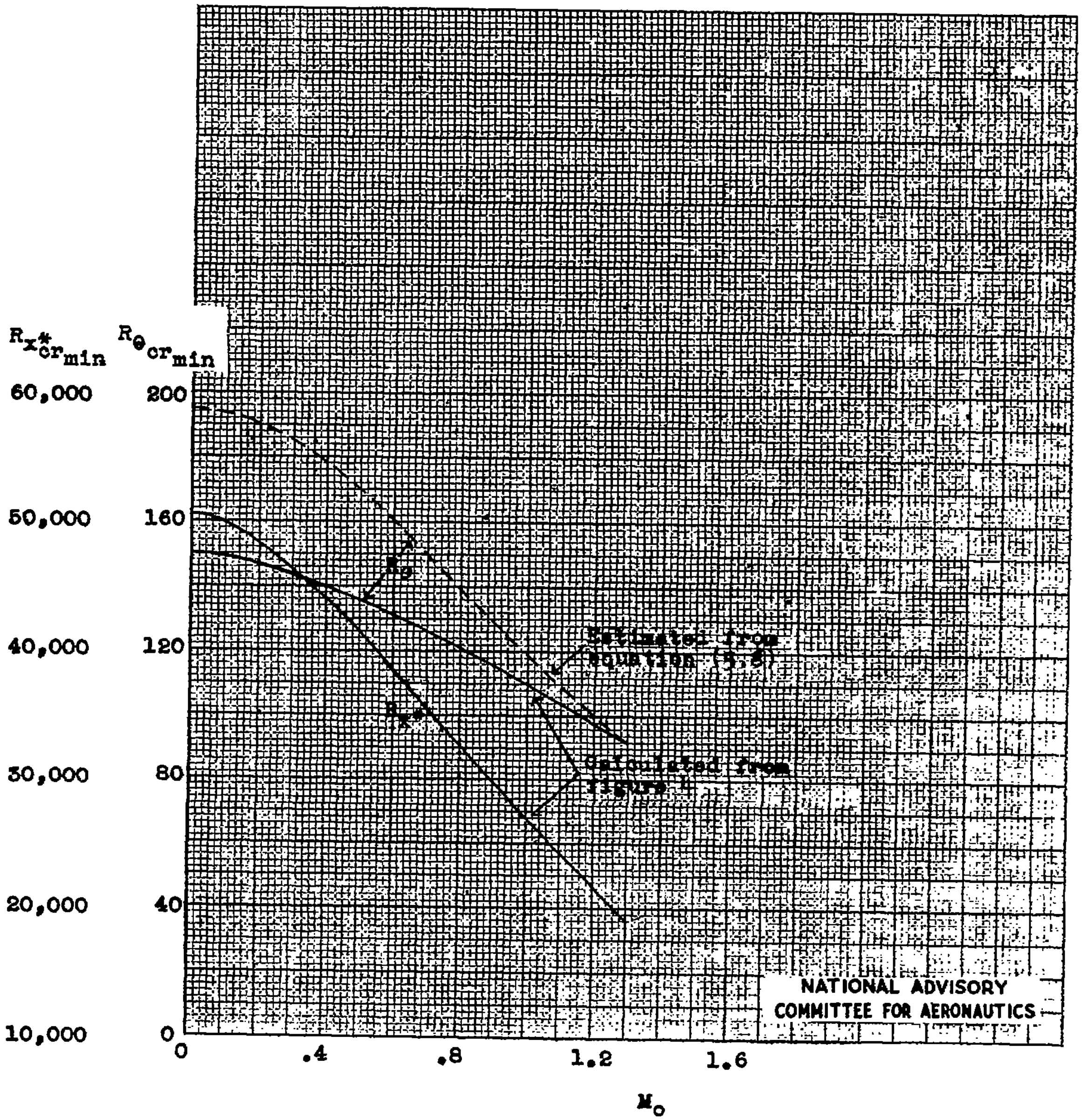
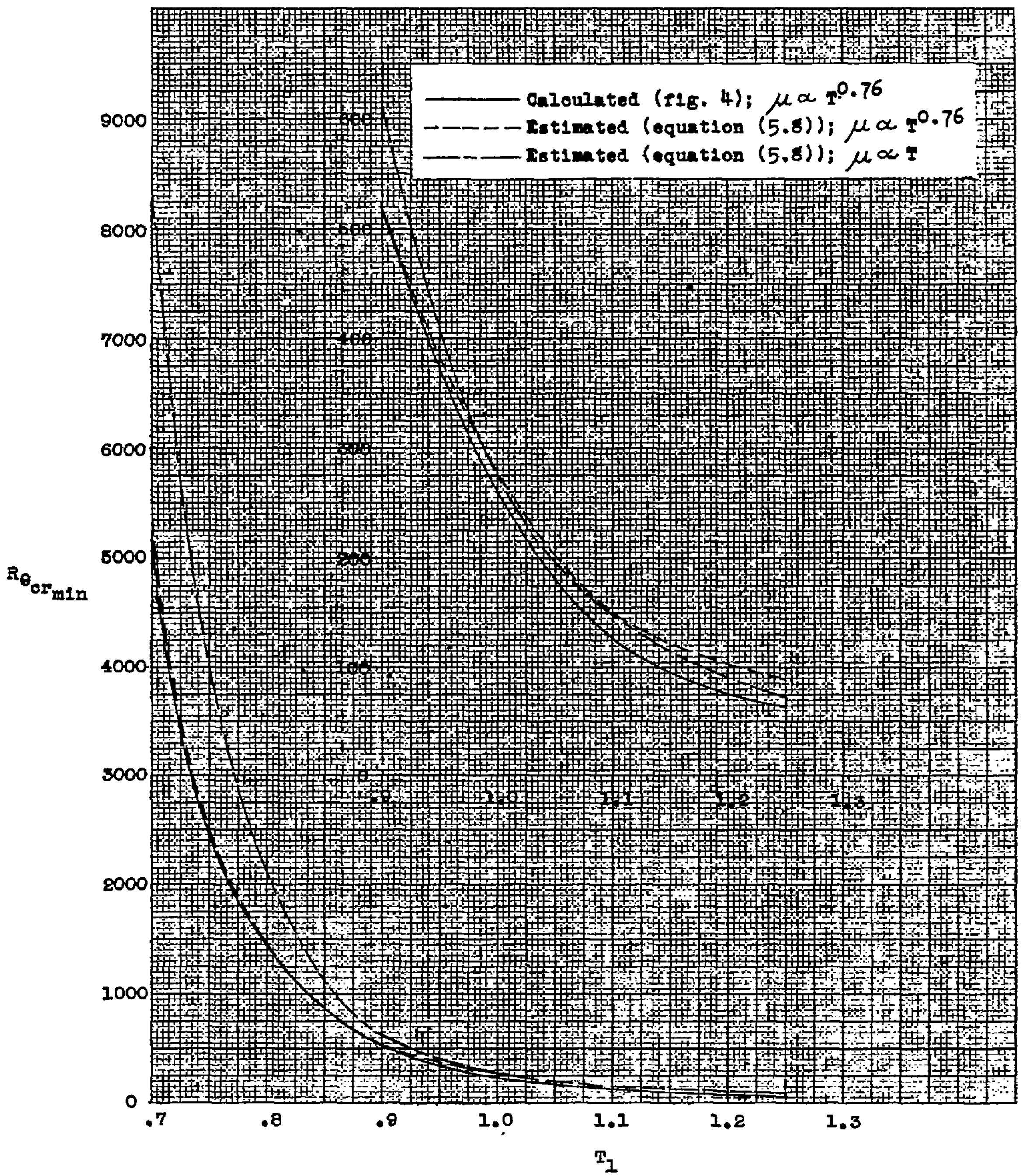


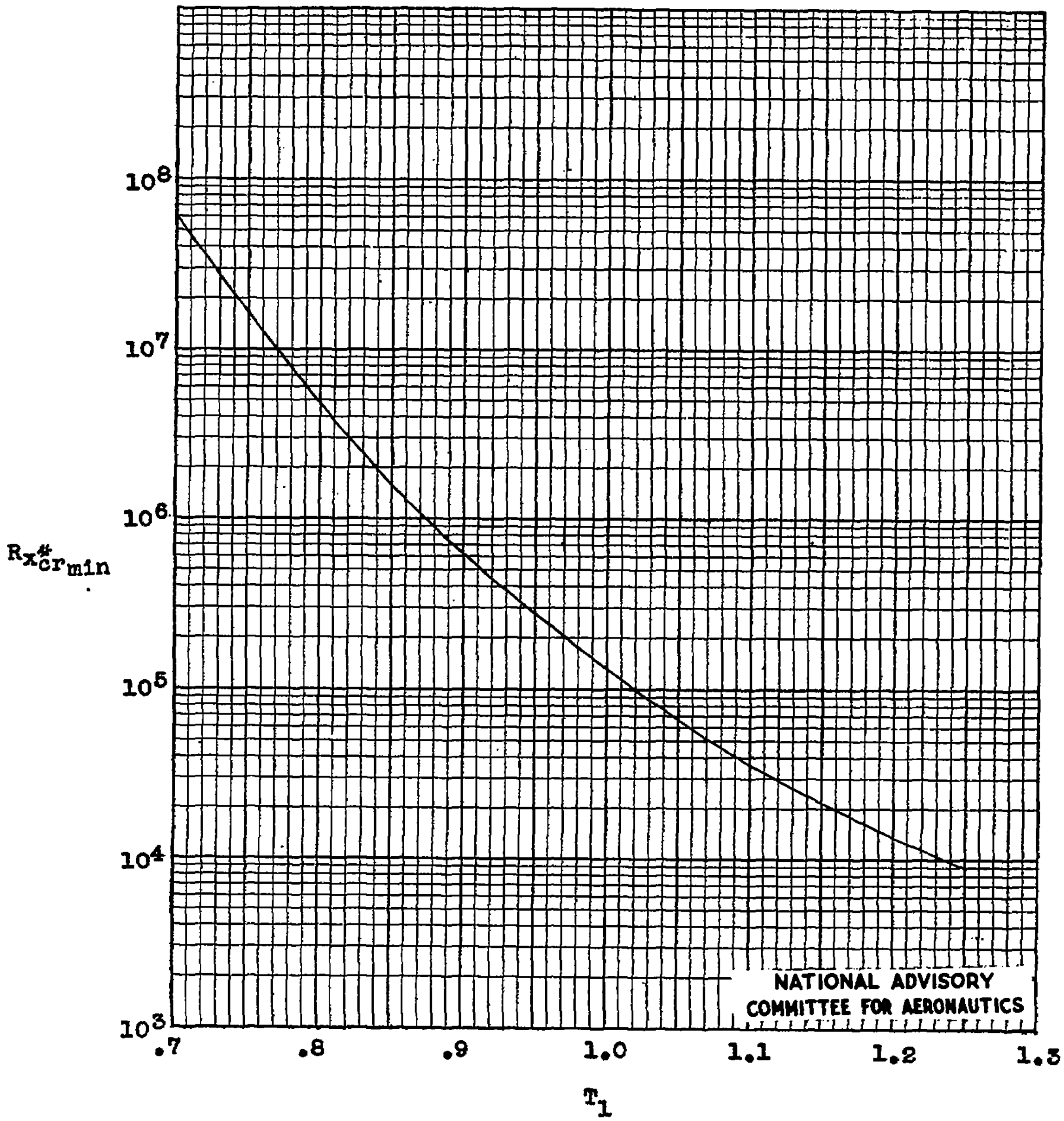
Figure 5.- Variation of minimum critical Reynolds number with Mach number for laminar boundary-layer flow. Insulated surface. $R_\theta = \left(\frac{\overline{u_o}^*\theta}{\nu_o^*}\right)$; $R_{x^*} = \left(\frac{\overline{u_o}^*x^*}{\nu_o^*}\right) = 2.25\ R_\theta{}^2$.



(a) $R_{\theta_{cr_{min}}}$ against T_1.

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

Figure 6.- Dependence of minimum critical Reynolds number on thermal conditions at solid surface. $M_o = 0.70$. T_1 is the ratio of surface temperature (deg abs.) to free-stream temperature (deg abs.).



(b) $R_{x^*_{cr_{min}}}$ against T_1 calculated from values of $R_{\theta_{cr_{min}}}$ taken from figure 4.

Figure 6.- Concluded.

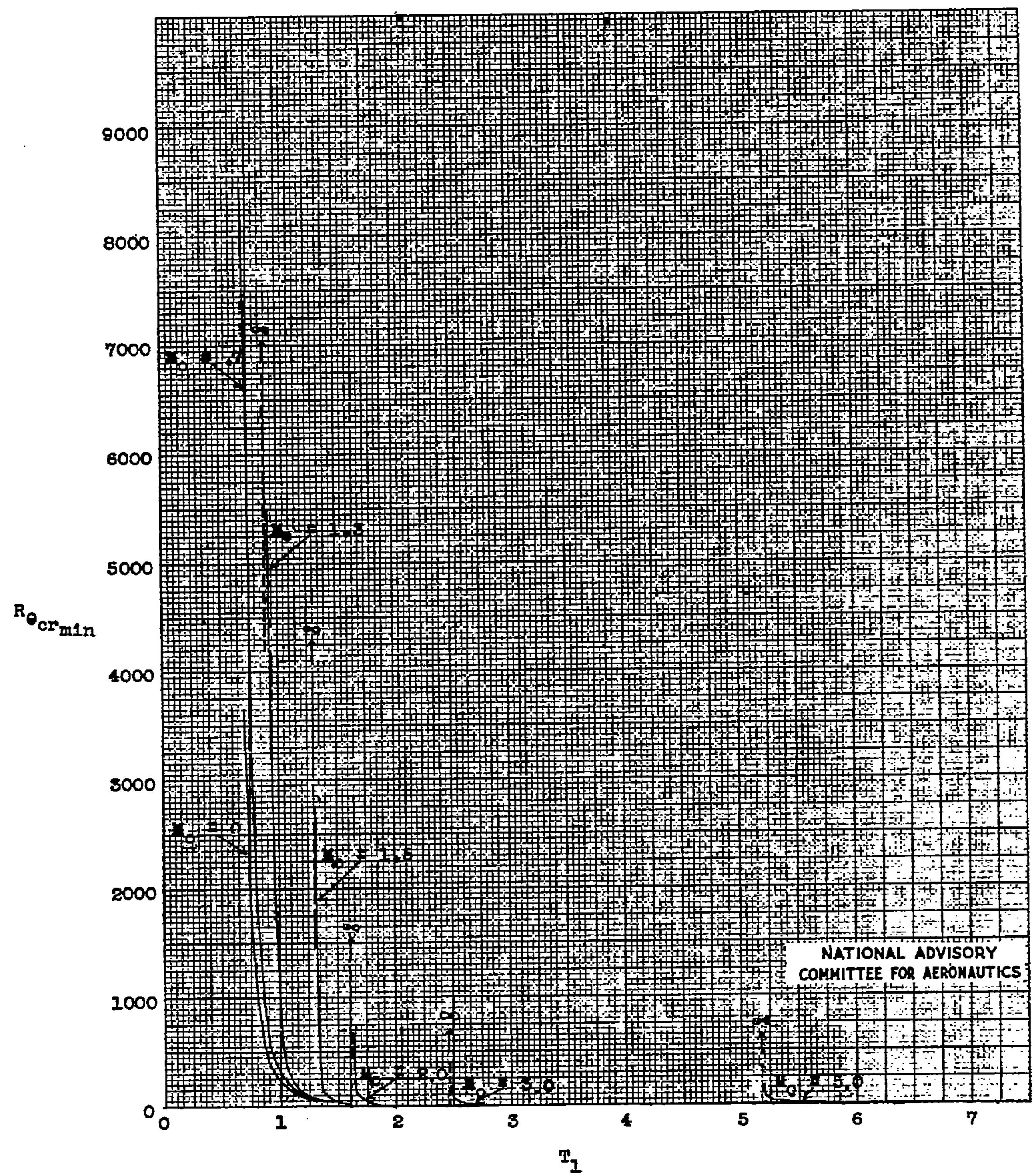


Figure 7.- Stabilizing effect at supersonic Mach numbers of withdrawal of heat from fluid through solid surface. At each value of $M_0 > 1$, there is a critical value of $T_1 = T_{1_{cr}}$ such that for $T_1 \leqq T_{1_{cr}}$ the laminar boundary-layer flow is stable at all values of the Reynolds number. (Curves for $M_0 = 0$ and $M_0 = 0.70$ included for comparison.) $R_{\theta_{cr_{min}}}$ estimated from equation (5.8). $\mu \propto T$; $\sigma = 1$.

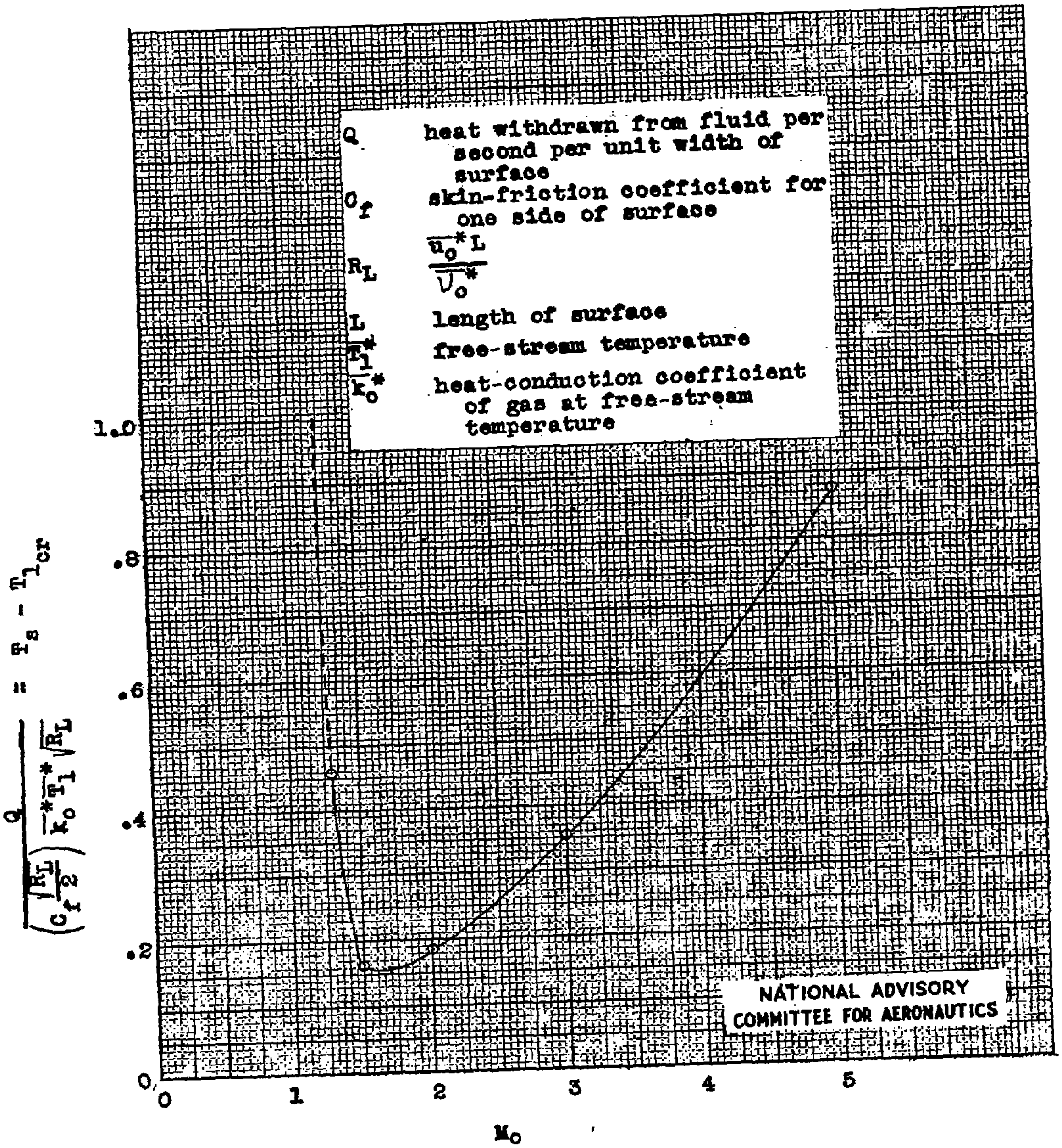


Figure 8.- Critical temperature ratio $T_{1_{cr}}$ for stability of laminar boundary layer against Mach number M_o. T_s is the ratio of stagnation temperature (deg abs.) to free-stream temperature (deg abs.) $= 1 + \frac{\gamma - 1}{2} M_o^2$ for $\sigma = 1$.

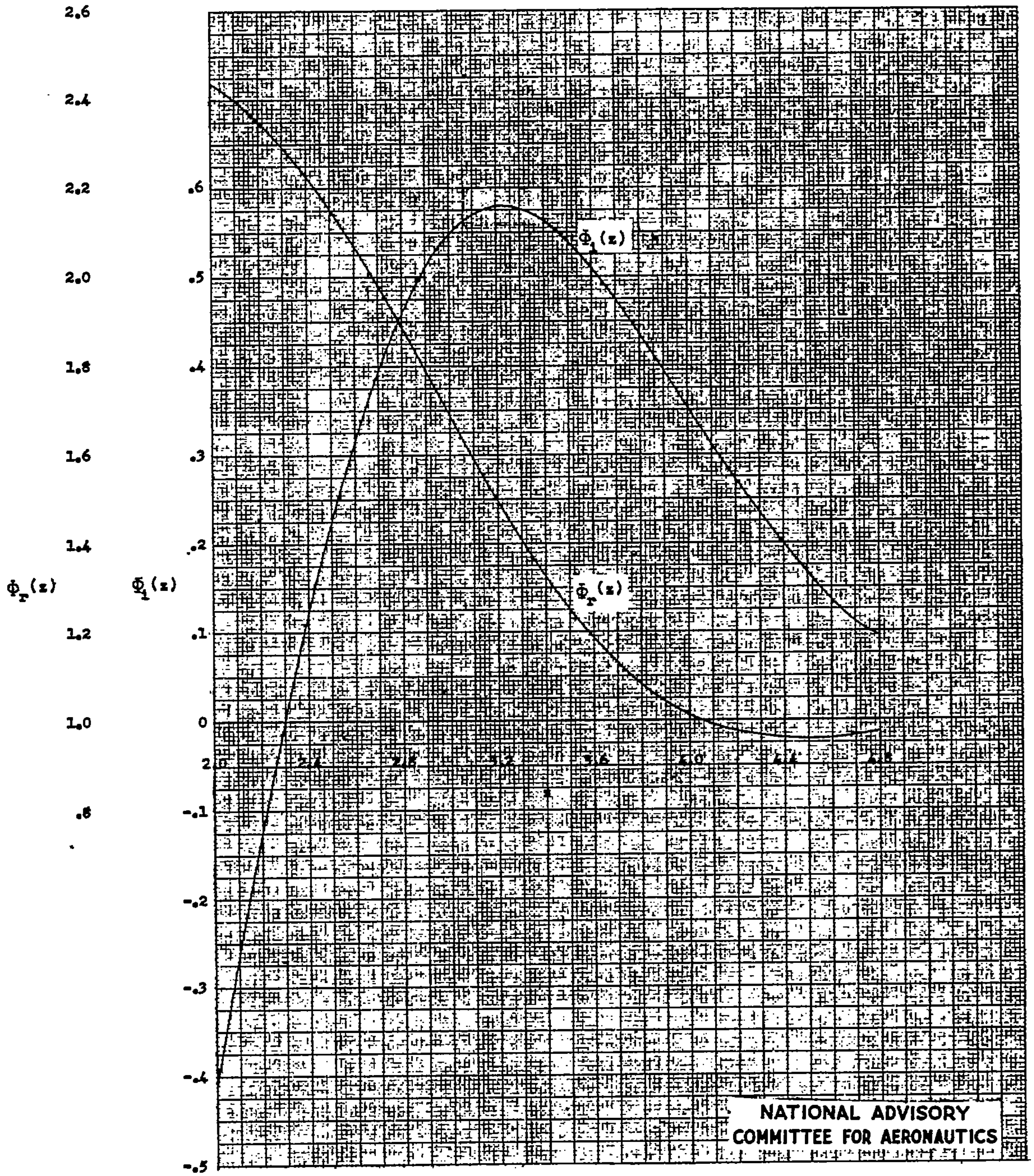


Figure 9.- The functions $\Phi_i(z)$ and $\Phi_r(z)$.